MISSOURI S&T™

Academic Notes

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Course Lecture Notes and Study Materials

BACHELOR OF SCIENCE, COMPUTER SCIENCE

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Preface

This compilation of academic notes and reference materials represents the intellectual foundation built during my studies at Missouri University of Science and Technology. These carefully organized notes capture key concepts, methodologies, and insights from across the computer science curriculum, serving as both a record of learning and a valuable reference resource.

From detailed lecture transcriptions to synthesized study guides, these materials demonstrate my systematic approach to learning and knowledge retention. The notes encompass theoretical foundations, practical applications, and problem-solving strategies that have proven invaluable throughout my academic career and continue to serve as a professional reference.

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"Miners Dig Deeper"

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CPE2210

Introduction To Computer Engineering

1 Test I Review

1.1 Redundancy Law

$$A + AB = A \tag{1}$$
$$A(A + B) = A \tag{2}$$

1 Digital Design

- nFET = Not Negated. Turns on by 1 or High.
- pFET = Negated. Turns on by 0 or Low.
- Xor = $\bar{A}B + A\bar{B}$
- Fan-out refers to the number of additional gates that can be physically connected to the output of a gate.
  - I_{ILMax} = The max current draw of a gate input for a low input value
  - I_{OLMax} = The max current output of a gate for a low output
  - I_{IHMax} = The max current draw of a gate input for a high input value
  - I_{OHMax} = The max current output of a gate for a high output
  - Max Fanout = $Min(\frac{I_{OLMax}}{I_{ILMax}}, \frac{I_{OHMax}}{I_{IHMax}})$
- We don't do POS for Variable Entered K-Maps. Turn into SOP.
- Inside Each Comparator
  - $out\_gt = in\_gt + (in\_eq * a * \overline{b})$
  - $out\_lt = in\_lt + (in\_eq * \bar{a} * b)$
  - $out\_eq = in\_eq * \overline{(a \oplus b)}$

CPE3150

Micro Embedded Design

3 Control Transfer Instructions and Time Delay Generation

- There are 5 types of instructions
  - One of which is `MOV, MOVC, MOVX`
- No one uses `JMP`.
  - Because it's a bit difficult to work with — as apposed to a long jump.
- `NOP` is literally do nothing
  - It's to kill time.
- Let's multiply 5×5

JZ Multiply By 5

```
        ORG 0

        MOV R0, #5
        MOV R1, #0

        MOV A, #0

LOOP:   MOV A, R1
        ADD A, #5

        MOV R1, A
        MOV A, R0

        DEC A
        JNZ A, LOOP

        END
```

DJNZ Multiply By 5

```
        ORG 0

        MOV R0, #5
        MOV A, #0

LOOP:   ADD A, #5
        DJNZ R0, Loop

        END
```

Nested Loop

Load `A` with 5, add 10 for 300 times.

```
        ORG 0

        MOV A, #5

        MOV R0, #3

LOOP2:  MOV R1, #100    ; Runs 3 times
LOOP1:  ADD A, #10      ; Runs 100 times
                        ; 3x100 = 300 times

        DJNZ R1, LOOP1
        DJNS R0, LOOP2
```

- 1 MC = $1.0852\mu s$ — **KNOW THIS**!
- For slides, the values (in order) of the delay is

Machine Cycle	**Times**
1	1
1	∞
2	∞
1	∞
2	∞
1	∞
2	$200 \times \infty$
2	$1 \times \infty$

Delay Example

```
DELAY:  MOV R0, #251

DL1:    MOV R1, #182
DL2:    DJNZ R1, DL2
        DJNZ R0, DL1

        RET
```

4 I/O Port Programming

- `Port 0` is weird, no pull up transistor
- `Port 0` also has bit addressability
- Only when ALE is set will the data be treated as an address `P0`
- `Port 1` general input/output port. Boring port.
- `Port 2` is date for input/output and addressing. Meaning it does not need pull up resistors.
- `Port 3` has special functions.
- For duty cycle.

$$f = 5\text{kHZ} \implies T = 200\mu s \tag{1}$$

$$\frac{200}{1.085} = 184\text{MC} \implies 92 \text{ iterations} \tag{2}$$

$$\frac{92}{5} = 18.4 \tag{3}$$

- I/0 Example — 2
  - We'll use `CJNE A, #` or `CJNE A, dir`
  - Also, `SUBB`

6 Arithmetic and Logic Instructions

- Unpacked = 4 0s out front
- Packed = no 0s out front, more effecient.
  - However, there are reasons for having 0000s out front
  - Could process nibble, instead of byte by byte
  - Makes processing easier
- BCD addition **does not** work.. kind of.
- `DA` takes into account the `AC` as well.
  - If `AC` $= 1$ or > 9, add $06H$
  - If `CY` $= 1$ or > 9, add $60H$.
- The extra `B` in `SUBB` mean subtracted with borrow.
  - Make sure to set clear `CY` before using `SUBB`
- One special case, divide by zero: `OV` $= 1$, values remain the same.
  - Division example: `A` $= 9,$ `B` $= 5$
- `XRL` only works for 8 bits.
  - Same addressing modes as for `ANL`
- Compliment works for `A`, `C` or *anything* that is bit addressable.
- **`CJNE` changes the CY flag**
- Serial Communication example (we use RLC because we want to use the Carry flag for transmitting data)

```
ORG 0

MOV A, #35H
MOV P2, #0
MOV R0, #8
```

```
    SETB P2.1
    SETB P2.1

TX: RLC A
    MOV P2.1, C
    DJNZ R0, TX

    SETB P2.1
    SETB P2.1

    END
```

- Same example, backwards

```
    ORG 0

    MOV R0, #8

RX: MOV C, P2.5
    RRC A
    DJNZ R0, RX

    MOV R2, A

    END
```

Number of '1"s example

```
    ORG 0
    MOV R0, #0  ; Counter for 1s
    MOV R1, #8  ; Counter for loop

    MOV A, P2

LP: RRC A

    DJNE
```

$$n\text{bit} \cdot n\text{bit} = 2n\text{bit} \tag{1}$$

```
ORG 0
MOV R0, #30H
MOV @R0, #0

XCHD A, @R0     ; M[30] = 07, A = 30H
SWAP A          ; A = 30H
ORL A, #30H
```

9 Timers and the 8051

- The time delay is $14 * 1.085 = 15.19\,\mu s$
- $65536 - \frac{7500}{1.085} = 58624_{10} = E500_{16}$
- $200 \cdot (65536 - 264) \cdot 1.085 = 14.16\,\mu s$

9 Timers

- `TF` is overflow flag, tells the timer is expired
  - `TR` stars/stops timer
- Math for programming timer in Mode 2: $\frac{1}{(241 \cdot 1.085 \cdot 10^{-6} \cdot 2)}$ slide 27

CS1200

Discrete Math

Theorem 5.3.1 For all integers

$$n0, 2^{2n} - 1$$

is divisible by 3.

Proof Let the property

$$P(n)$$

be the sentence

$$“2^{2n} - 1 is divisible by 3.''$$

$$2^{2n} - 1 is divisible by 3.$$

**Show that

$$P(0)$$

is true**: To establish

$$P(0)$$

, we must show that

$$2^{2*0} - 1$$

is divisible 3.

But

$$2^{2*0} - 1 = 2^0 - 1 = 1 - 1 = 0$$

and 0 is divisible by 3 because

$$0 = 3 * 0$$

. Hence

$$P(0)$$

is true.

**Show that for all intergers $k0$, if $P(k)$ is true then $P(k+1)$ is also true.

Problem: Prove by mathematical induction that $\forall n \in \mathbb{Z}^+ \cup \{0\}, 6|7^n - 1$

Solution: Proof [*By Mathematical Induction*]. Let the predicate $P(k)$ be the statement $6|7^n - 1$. We will prove that $P(n)$ is true $\forall n \in \mathbb{Z}^+ \cup \{0\}$.

Basis Step: Show that $P(0)$ is true. $P(0)$ is true because $6|7^0 - 1 = 6|1 - 1 = 6|0$

Inductive Step: Show that $\forall n \in \mathbb{Z}^+ \cup \{0\}, P(k) \rightarrow P(k+1)$. Let $k \in \mathbb{Z}^+ \cup \{0\}$ and suppose that $6|7^k - 1$ (inductive hypothesis.) Show that $6|7^{k+1} - 1$. By definition of divisibility, the inductive hypothesis is equivalent to $\exists r \in \mathbb{Z}, 7 - 1 = 6r$

Then $7^{k+1} - 1 = 7 * 7^K - 1$
$= 7 * 7^k - 1$
$= (6+1)7^k - 1$
$= 6 * 7^k + (7^k - 1)$
$= 6 * 7^k + 6r$
$= 6(7^k + r)$

Now $7^k + r \in \mathbb{Z}$ because the products and sums of integers are integers. Thus by divisibility, $6|7^{k+1} - 1$.

Problem: A sequence d_1, d_2, d_3 is defined by letting $d_1 = 2$ or $\forall k \in \mathbb{Z}, k2.d_k = \frac{d_k - 1}{k}$ Prove that $\forall n \in \mathbb{Z}^+, d_n = \frac{2}{n!}$

Solution: Proof (By mathematical induction) According to the definition of $d_1, d_2, d_3, \cdots$, we have $d_1 = 2$ or $\forall k \in \mathbb{Z}, k2, d_k \frac{d_k - 1}{k}$. Let the predicate be the equality $d_n = \frac{2}{n!}$ We will prove that $P(n)$ is true $\forall n \in \mathbb{Z}^+$

Basis Step: Show that $P(1)$ is true. Left hand side of $P(1)$ is $d_1 = 2$, right hand side of $P(1)$ is $\frac{2}{1!} = 2$, so left hand side and the right hand side of $P(1)$ are equal, so $P(1)$ is true.

Inductive Step: Show $\forall k \in \mathbb{Z}^+, P(k) \rightarrow P(k+1)$. Let $k \in \mathbb{Z}^+$ and suppose that $d_k = \frac{3}{k!}$. We must show that $d_{k+1} = 2(k+1)!$

Left hand side of $P(k+1)$ is $d_{k+1} = \frac{d_k}{k+1}$
$= \frac{\frac{2}{k!}}{k+1}$
$= \frac{2}{k!(k+1)}$
$= \frac{2}{(k+1)!}$

Problem: Prove by Mathematical Induction $\forall n \in \mathbb{Z}, n2, 5^n + 9 < 6^n$ **Answer**: Proof by Mathematical Induction. Let the predicate $P(n)$ be the statement $5^n + 9 < 6^n$. We will prove that $P(n)$ is true $\forall n \in \mathbb{Z}, n2$.
Basis Step; Show that $P(2)$ is true because the left hand side is $5^2 + 9 = 25 + 9 = 34$ and the right hand side is $6^2 = 36$ and $34 < 36$.

Inductive Step: Show that $\forall k \in \mathbb{Z}, k2, P(k) \rightarrow P(k+1)$. Let $k \in \mathbb{Z}, k2$ and suppose $5^k + 9 < 6^k$ (The inductive hypothesis) We must show that $5^{k+1} + 9 < 6^{k+1}$.

Multiply both the sides of of the inductive hypothesis with 5 yields $5(5^k + 9) < 5 * 6^k$.
Note that $5^{k+1} + 9 < 5^{k+1} + 45 = 5(5^k + 9)$ and $5 * 6^k < 6^{k+1}$.
Thus: $5^k + 1 + 9 < 5(5^k + 9)$ and $5(5^k + 9 < 5 * 6^k$ and $5 * 6^k < 6 * 6^k = 6^{k+1}$

Thus, by transitivity, $5^{k+1} < 6^{k+1}$

1 Chapter 6.1: Set Theory

Problem 5.13: Prove by Mathematical induction $\forall n \in \mathbb{Z}^+ \cup \{0\}, x, y \in \mathbb{Z}, x \neq y, x - y | x^n - y^n$

Proof Suppose $x, y \in \mathbb{Z} \vee x \neq y$. Let the predicate $P(n)$ be the statement $x - y | x^n - y^n$.

Basis Step Show that $P(0)$ is true. $P(0)$ is true because $x^0 - y^0 = 1 - 1$ and $x - y | 0$.

Inductive Step Show that $\forall k \in \mathbb{Z}^+ \cup \{0\}, P(k) \to P(k+1)$

Let $k \in \mathbb{Z}^+ \cup \{0\}$ and suppose $x - y | x^k - y^k$. We must show that

$$x - y | x^{k+1} - y^{k+1}$$

. By rewriting the inductive hypothesis using the definition of divisiblity, we have:

$$\exists r \in \mathbb{Z}, x^k - y^k = (x - y)r$$

Then:

$$x^{k+1} - y^{k+1}$$

=

$$x^{k+1} - xy^k + xy^k - y^{k+1}$$

=

$$x(x^k - y^k) + y^k(x - y)$$

=

$$x(x - y)r + y^k(x - y)$$

=

$$(x - y)[xr + y^k]$$

Now

$$x(r + y^k) \in \mathbb{Z}$$

because products and sums of integeres are integers. Thus,

$$x - y | x^{k+1} - x^{k+1}$$

.

Let the set

$$A = \{x \in \mathbb{Z} | x = 5a + 2, a \in \mathbb{Z}\}$$

$$B = \{y \in \mathbb{Z} | y = 10b - 3, b \in \mathbb{Z}\}$$

$$C = \{z \in \mathbb{Z} | z = 10c + 7, z \in \mathbb{Z}\}$$

A. Is $A \subset B$? . No, because 2 = 5 * 0 + 2, but $2 \not\subset B$ because if $2 \in B$ then $\exists b \in \mathbb{Z}$ such that $2 = 10b - 3$. Then $10b = 5, b = \frac{1}{2}$ which is not an integer. Counterexample found.

B. Is $B \subset A$? . Yes. Proof: Suppose $y \in B$. [*We must show that $y \in A$. By definition of A, this means that we must show $y = 5(\text{some int}) + 2$]. By definition of B, $\exists b \in \mathbb{Z}, y = 10b - 3$. Let $a = 2b - 1$. Then $a \in \mathbb{Z}$ and $5a + 2$ =

$$5(2b - 1) + 2 = 10b - 5 + 2 = 10b - 3 = y$$

To prove A = B, prove both are subsets of each other Let the universal set be $\mathbb{R}$

$$A \cup B = \{x \in \mathbb{R} | 6 < x \leq 8\}$$

$$A \cap B = \{x \in \mathbb{R} | -1 < x \leq 0\}$$

$$A^c = \{x \in \mathbb{R} | -3 > x \text{ or } x > 0\}$$

$$A \cap C = \emptyset$$

Draw the venn diagram to describe $A \cap B \neq \emptyset, B \cap C = \emptyset, A \cap C = \emptyset, A \not\subset B, C \not\subset B$

Questions: Is $0 \in \emptyset$? No $\emptyset \neq \{\emptyset\}$? Yes. $\emptyset \in \{\emptyset\}$ $\emptyset \notin \emptyset$

EXAM QUESTION

$$\forall i \in \mathbb{Z}^+, R_i = \{x \in \mathbb{R} | 1 \leq x \leq 1 + \frac{1}{i}\} = [1, 1 + \frac{1}{i}]$$

$$\cup_{i=1}^4 R_i = R_1 \cup R_2 \cup R_3 \cup R_4 = \cup [1, 1 + \frac{1}{1}] \cup [1, 1 + \frac{1}{2}] \cup [1, 1 + \frac{1}{3}] \cup [1, 1 + \frac{1}{4}] = [1, 2]$$

$$\cap_{i=1}^4 R_i = R_1 \cap R_2 \cap R_3 \cap R_4 = \cap [1, 1 + \frac{1}{1}] \cap [1, 1 + \frac{1}{2}] \cap [1, 1 + \frac{1}{3}] \cap [1, 1 + \frac{1}{4}] = [1, 1 + \frac{1}{4}]$$

Is $\{\{w, x, v\}, \{w, y, q\}, \{p, z\}\}$ a partition of $\{p, q, u, v, w, x, y, z\}$? Yes.

Suppose A = {1,2}, B = {2,3}
b. $P(A) = P(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
d. $P(A \times B) = P(\{(1,2),(1,3),(2,2),(2,3)\})$

$$= \{\emptyset, \{(1,2)\}, \{(1,3)\}, \{(2,2)\}, \{(2,3)\}, \tag{1}$$
$$\{(1,2),(1,3)\}, \{(1,2),(2,2)\}, \{(1,2),(2,3)\}, \ldots\} \tag{2}$$

you get the idea.
a. $P(\emptyset) = \{\emptyset\}$
b. $P(P(\emptyset)) = P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$
c. $P(P(P(\emptyset))) = P(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
Let A = {a,b}, B = {1,2}, C = {2,3} $A \times (B \cap C) = \{a,b\} \times \{2\} = \{(a,2),(b,2)\}$
x = 1,3,5, y = a, b, c, d g: x → Y defined by 1 = b; 3 = b; 5 = b;
a. domain of g: 1, 3, 5 co-domain of g:
b. g(1) = b g(2) = b g(3) = b
range of g: is 3 an inverse image of a? Nah is 3 an inverse image of b? Yes
What is the inverse image of b? 1,3,5 What is the inverse image of c? $\emptyset$
Represent g as a set of ordered pairs: (1,3), (3,b), (5,b)
Find all functions from X = a,b,c to Y = u; ¡ATTACH SKITCH¿
Find all the functions from X = a,b,c to Y u, v
Let F and G be the functions from ℝ to ℝ Define new functions F - G: ℝ → ℝ and G - F: ℝ → ℝ as: $\forall x \in \mathbb{R}$, $(F - G)(x) = F(x) - G(x)$ $(G - F)(x) = G(x) - F(x)$
Does $F - G = G - F$ No.

Let $x = a, b, c$. Define relation **J** on $P(x)$ as follows: $\forall A, B \in P(x), A\mathbf{J}B \Leftrightarrow A \cap B \neq \emptyset$
a) Is $\{A\}\mathbf{J}\{C\}$? No, because $\{A\} \cup \{C\} = \emptyset$
b) Is $\{a, b\}\mathbf{J}\{b, c\}$? Yes, because $\{a, b\} \cap \{b, c\} = \{b\} \neq \emptyset$
c) Is $\{a, b\}\mathbf{J}\{a, b, c\}$? Yes, because $\{a, b\} \cap \{a, b, c\} = \{a, b\} \neq \emptyset$

Let $A = \{3, 4, 5\}, B = \{4, 5, 6\}$ and let **S** be the "Divides" relation: $\forall (x, y) \in A \times B, x\mathbf{S}y, \Leftrightarrow x|y$
a) $S = \{(3, 6), (4, 4), (5, 5)\}$
b) $S^{-1} = \{(6, 3), (4, 4), (5, 5)\}$

Suppose a function $F : X \to Y$ is onto but not one-to-one. Is F^{-1} a function? No, because if $F : X \to Y$ is not one-to-one, then $\exists (x_1, x_2) \in X \bigwedge \exists y \in Y, x_1 \neq x_2 \bigwedge (x_1, y) \in F \bigwedge (x_2, y) \in F$. But this implies $\exists x_1, x_2 \in X \bigwedge \exists y \in Y, x_1 \neq x_2 \bigwedge (y, x_1) \in F^{-1} \bigwedge (y, x_2) \in F^{-1}$. Consequently, F^{-1} does not satisfy property (2) of the definition of function.

Define relations **R** and **S** on $\mathbb{R}$ as follows:
$\mathbf{R} = \{x, y\} \in \mathbb{R}x\mathbb{R} | x^2 + y^2 = 4$ Square!
$\mathbf{S} = \{x, y\} \in \mathbb{R}x\mathbb{R} | x = y$ Line!
Graph $R, S, R \cup S, R \cap S$ in the Cartesian plane. *R, Circle. S, Line. R and S, both of them. R or S, just the intersection points.*

Let $A = \{2, 3, 4, 6, 7, 9\}$ and define a relation $\mathbb{R}$ on A as follows: $\forall x, y \in A, x\mathbf{R}y \Leftrightarrow 3|(x - y)$.
1. Each element has a loop to itself.
2. If element a is related to element b, then b is also related to a.
3. In each case where there is an arrow from one point to another point, and that to a third, there is an arrow going from the first to the third.

1. Reflixive? Yes.
2. Symmetric? HELL YES.
3. Transitive? HELL TO THE HELL TO THE YES

R is reflexive iff $\forall x \in A, x\mathbf{R}x$ R is symmetric iff $\forall x, y \in A, x\mathbf{R}y \to x\mathbf{R}y$ R is transitive iff $\forall x, y, z \in A, x\mathbf{R}y \bigwedge y\mathbf{R}z \to x\mathbf{R}z$

Let $A = \{0, 1, 2, 3\}$ and define relations **R, S, T** as follows:
R: $\{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$
S: $\{(0, 0), (0, 2), (0, 3), (2, 3)\}$
T: $\{(0, 1), (2, 3)\}$
a) Is **R** reflexive, symmetric, transitive? Yes, Yes, Nah.
b) Is **S** reflexive, symmetric, transitive? No, no, yasss.
c) Is **T** reflexive, symmetric, transitive? No, no, yasss.

Reflexivity, Symmetry, Transitivity

Properties of Congruence Modulo 3

Define relation T on $\mathbb{Z}$ as follows: $\forall m, n \in \mathbb{Z}, m\ T\ n \Leftrightarrow 3 \mid (m - n)$

1. **Reflexive?** Yes. Proof: Let $m \in \mathbb{Z}$. We must show m T m. Now *m - m = 0*. But *3 — 0* since 0 = 3 × 0. Hence *3 — (m - m)*. Thus by definition of T, $m\ T\ m$.

2. **Symmetric?** Yes. Proof: Left $m, n \in \mathbb{Z}, mTm$. We must show that $n\ T\ m$. By definition of T. Since $m\ T\ n$, then *3 — (m - n)*. Dy definition of divides, $\exists k \in \mathbb{Z}, m - n = 3k$. Multiplication on both sides by *-1* gives $n - m = 3(-k)$. Since $-k \in \mathbb{Z}, 3|(n - m)$. Hence, by definition of $T, n\ T\ m$.

3. **Transitive?** Hell to the yes. Proof: Let $m, n, p \in \mathbb{Z}, m\ T\ n \bigwedge n\ T\ p$. We must show $m\ T\ p$. By definition of T, since $m\ T\ n \bigwedge n\ T\ p$, then $3|(m - n) \bigwedge 3|(n - p)$. By definition of divides, $\exists r \in \mathbb{Z}, m - n = 3r \bigwedge \exists s \in \mathbb{Z}, n - p = 3s$ Adding the two equations gives $(m - n) + (n - p) = 3r + 3s$ and simplifying gives $m - p = 3(r + s)$. Since $r + s \in \mathbb{Z}, 3|(m - p)$. Hence, by definition of T, $m\ T\ p$.

Transitive Closure

Let A be a set and R a relation on A. The Transitive Closure of R is the relation R^t on A that satisfies the following three properties:

1. R^t is transitive.

2. $R \subset R^t$.

3. If S is any other transitive relation that contains R, then $R^t \subset S$

Example

Let $A = \{0, 1, 2, 3\}$ and Relation R on A is $R = \{(0, 1), (1, 2), (2, 3)\}$. Find the transitive closure of R.
Answer: $\{(0, 1), (1, 2), (2, 3)\} \subset R^t . R^t = \{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}$

Equivalent Relations

Partition

A partition of set A is a set of mutually disjoint **non-empty** subsets of A whose union is A.

Give a partition of set A, the relation induced by the partition, R, is defined as follows: $\forall x, y, \in A, x\ R\ y \Leftrightarrow \exists$ subsets Ai of the partition, $x, y \in Ai$.

Example

Let $A = \{0, 1, 2, 3, 4\}$ and consder the following partition of A: $\{0, 3, 4\}, \{1\}, \{2\}$.
Find the relation R induced by this partition.
Answer: $R = \{(0, 0), (0, 3), (0, 4), (1, 1), (2, 2)\}$

Theorem

Let A be a set with partition and let R be the relation induced by the partition. Then R is *reflexive, symmetric, and transitive.*

Proof: Suppose A is a set with partition $\{A_1, A_2, \ldots, A_n\}$. Without loss of generality, we'll assume a finite partition. Then $\forall i, j \in \{1, 2, \ldots, n\}, i \neq j \Rightarrow A_i \cap A, = \not\emptyset$. And $\bigcup_{i=1}^{n}$. The relation R induced by the partition is defined as follows: $\forall x, y \in A, x\ R\ y \Leftrightarrow \exists Ai$ of the partition, $x, y \in Ai$.

Reflexive Let $x \in A$, Since $A_1, A_2, \ldots A_n$ is a partition of A, $\exists i \in \{1, 2, \ldots n\}, x \in Ai$. But then the statement $\exists Ai$ of the partition, $x \in Ai$ and $x \in Ai$ is true. Thus, by definition of R, $x\ R\ x$.

Symmetric Let $x, y \in A, x\ R\ y$. Then $\exists Ai$ of the partition, $x \in Ai \wedge y \in Ai$. It follows that the statement $\exists Ai$ of the partition, $y \in Ai$ is true. Hence, by definition of R, $x\ R\ y$.

Transitivity Let $x, y, z \in A, xRy \wedge yRz$. By definition of R, $\exists Ai, A,$ of the partition $x, y \in A \wedge y, z \in A$. Suppose $Ai \neq A$, then $Ai \cap A, = \not 0$. Since $\{A, A_2, \ldots, A_n\}$ is a partition of A. But $y \in Ai \bigwedge y \in Aj,$. Hence $Ai \cap Aj\ \neq \not 0$. That's a contradiction, thus Ai = A. It follows that $x, y, z \in Ai$. Thus, by definition of R, xRz.

Let A be a set and R a relation on A, R is an equivalence relation iff R is reflexive, symmetric, and transitive.

1 Partial Order Relations

1.1 Antisymmetry

No symmetry, **at all.** Let R be a relation on a set A. R is **antisymmetric** iff $\forall a, b \in A, a\ R\ b \bigwedge b\ R\ a \to a = b$.

1.2 Partial Order Relations

Let R be a relation defined on a set A. R is a **partial order relation** if, and only if, R is reflexive, antisymmetric, and transitive.

1.2.1 Example

Let P be the set of all people who have everlived and define A relation $P\ on\ P$ as follows: $\forall r, s \in P, r\ R\ p \Leftrightarrow$ r is an ancestor of s or $r = s$. Is R a partial order relation?

Yes. Proof.

R is reflexive Suppose $r \in P$. Then $r = r$, by the definition of R, $r\ R\ r$

R is antisymmetric Suppose $r, s \in P, s \in P \bigwedge r\ R\ s \bigwedge s\ R\ r$. We must show that $r = s$. By definition of R, either r is an ancestor of s, or $r = s$, and also, either s is an ancestor of r, or $s = r$. Now it impossible for both r to be an ancestor of s and for s to be an ancestor of r. Hence of the conditions must be false, and so $r = s$.

R is transitive Suppose $r, s, t \in P \bigwedge r\ R\ s \bigwedge s\ R\ t$. We must show that $s\ R\ t$. By definition of R, either r is an ancestor of s, or $r = s$ and either s is an ancestor of t, or $s = t$.

case 1 In case r is an ancestor of s is an ancestor of t, then r is an ancestor of t, and so $r\ R\ t$.

case 2 In case r is an ancestor of s and $s = t$, then r is an ancestor of t, and so $r\ R\ t$.

$\vdots$

case 3 $\cdots$

case 4 $\cdots$

1.2.2 Example

Define a relation r on $\mathbb{Z}$ as follows: $\forall m, n \in \mathbb{Z}, m\ R\ n \Leftrightarrow$ ever prime factor of m is a prime factor of n.

No, because it's not antisymmetric not antisymmetric. Counter example: let $m = 2 \bigwedge n = 4$. Then $m\ R\ n$ because every prime factor of *2* is a prime factor of 4, and $n\ R\ m$ because 4 is a prime factor of 4. But $m \neq n$, because $2 \neq 4$

1.3 Dictionary or Lexicographic

Let A be a set with a partial order relation R, and let S be a set of strings over A. Define a relation $\preceq$ on S as follows:

For any two strings in S, $a_1 a_2 \cdots a_m$ and $b_1 b_2 \cdots b_n$, where m and n are positive integers,

1. $m \leq n$ $a_i = b_i$ for all $i = 1, 2, \ldots, m$, then

$$a_1 a_2 \cdots a_m \preceq b_1 b_2 \cdots b_n.$$

2. If for some integer k with $k \leq m, k \leq n,$ and $k \geq 1, a_i = b_i$ for all $i = 1, 2, \ldots, k-1$ and $a_k \neq b_k$, but $a_k\ R\ b_k$ then.

$$a_1 a_2 \cdots a_m \preceq b_1 b_2 \cdots b_n.$$

3. If ε is the null string and s in any string in S, then $\epsilon \preceq s$.

If no strings are related other than by these three conditions, then $\preceq$ is a partial order relation.

1.3.1 Example

Let $A = \{a, b\}$ and suppose A has the partial order relation $R = \{(a, a), (a, b), (b, a), (b, b)\}$. Let S be the set of all strings in a's and b's and let $\preceq$ be the corresponding lexiographic order on S.

C: $\varepsilon \preceq aba$? Yes, property #3.

E: $bbab \preceq bbaa$? No, property #2.

F: $ababa \preceq ababaa$? Yes, property #1.

9 Counting And Probability

9.1 Introduction

Example 9.1.12

a) List the eight possibilites: $\{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$
b) $E_i = \{BBG, BGB, GBB\}$. Chance of of one being girl: $\frac{3}{8}$.
$E_2 = \{BGG, GBG, GGB, GGG\}$. Chance of being *at least* two girls: $\frac{1}{2}$

Example 9.1.13

The monty hall problem.. Woo.

Theorem 9.1.1

$m, n \in \mathbf{Z} \bigwedge m \leq n \Rightarrow$ There are textitn - m + 1 integers from m to n inclusive.

Exercise 9.2.5

Creating Trees.

Example 9.2.15

A combination lock requires 3 selections of numbers 1 - 30.

- How many combinations? 30^3
- How many without repeats? $30 * 29 * 28$

Example 9.2.27

```
for i = 5 to 50
    for j = 10 to 20
        myFunction();
```

How many function calls? 506.

Example 9.2.3

- How many ways to rearrange the letters in Algorithm? $\frac{9!}{6!}$

9.3 Counting Elements of Disjoint Sets: The Addition Rule

0.1 The Addition Rule

Let $\{A_1, A_2, \ldots, A_k\}$ be a partition of A, then $N(A) = N(A_1) + N(A_2) + \ldots, N(A_k)$

0.1.1 Example 9.3.7

In some state all license plates consist of 4-6 symbols chosen form 26 letters and 10 digits.

- How many license plates without repetition? $36^4 + 36^5 + 36^6$
- No repetition? $\frac{36!}{32!} + \frac{36!}{31!} + \frac{36!}{30!}$
- How many license plates have repition? $36^4 + 36^5 + 36^6 - [\frac{36!}{32!} + \frac{36!}{31!} + \frac{36!}{30!}]$
- What is the probability that a random license plate has a repeat?

$$\frac{36^4 + 36^5 + 36^6 - [\frac{36!}{32!} + \frac{36!}{31!} + \frac{36!}{30!}]}{36^4 + 36^5 + 36^6} \tag{1}$$

0.2 The Difference Rule

If $B \subset A \Rightarrow N(A - B) = N(A) - N(B)$

Example 9.3.9b

```
for i = 0 to 4
    for j = 1 to i
        printf("Hello, world");
```

How many runs? $\frac{n(n+1)}{2}$.

0.2.1 Probability of the Complement of an Event

If S is a finite sample space and A is an event in S, then:

$$P(A^c) = 1 - P(A)$$

Let A = *Everyone Aces Exam III*,
A^c = *not everyone aces exam III.*
$P(A) = 1\%$
$P(A^c) = 1 - 1\% = 99\%$

Assuming all yeas have 365 days and all birthdays occur with equal probability, how large must n be so that in any randomly chosen group of n people, the profitability that two or more have the same birthday is $\geq \frac{1}{2}$

- Probability that no two people have same birthday $= \frac{365 \times 364 \times \ldots \times 365 - n + 1}{365^n}$
- Probability that two or more have same birthday $= 1 - \frac{365 \times 364 \times \ldots \times 365 - n + 1}{365^n}$

A college conducted a survey to explore academic interests of students. If asked students to place checks besides the following statements if they were true:

1. I was on the honor roll last term.
2. I belong to an academic club.
3. I'm majoring in multiple subjects.

Out of 100 students, 28 checked #1, 26 checked #2, and 14 checked #3, #8 checked both #1 and #2, 4 checked #1 and #3, 3 checked #2 and #3, and 2 checked all three.

- How many checked at least one? $N(H \wedge C \wedge \wedge D) = N(H) + N(C) + N(D) - N(N \wedge C) - N(H \wedge D) - N(C \wedge D) + N(H \wedge C \wedge D) =$ **28 + 26 + 14 - 8 - 4 - 3 + 2** $=$ **55**
- How many checked none? **100 - 55 = 45**
- Let H be the set of students who checked #1, C those who checked #2, and D those who checked #3. *Insert Picture*
- How many checked #1 and #2, but not #3. $N(H \wedge C) - N(H \wedge C \wedge D) =$ **8 - 2 = 6**
- How many checked #2 and #2, but not #1. $N(C \wedge D) - N(C \wedge D \wedge H) =$ **3 - 2 = 1**
- How many checked #2, but neither of the others? $N(C) - N(C \wedge H) - N(C \wedge D) + N(C \wedge D) + N(C \wedge H \wedge D) =$ **26 - 8 - 3 + 2 = 17**

1 8.4 The Pigeonhole Problem

- If 13 cards are selected from a standard 52-card deck, must at least 2 be of the same denomination? No, example: $2, 3, 4, \ldots, 10, J, Q, K, A$.
- If 20 cards are selected from a standard 52-card deck, must at least 2 be of the same denomination? Yes, let X be the set consisting of the 20 selected cards and let Y be the 13 possible denominations. Define a function D from X (pigeons) to Y (Pigeonholes) by specifying $\forall x \in X, D(X) = The denomination of X$. Now X has 20 elements and Y has 13, and $20 > 13$. So by the pigeonhole principle, D is not one-to-one. Hence, $\exists x_1, x_2, x_1 \neq x_2 \wedge D(x_1) = D(x_2)$. Then x_1 and x_2 are distinct cards that have the same denomination.

Example 4.27

In a group of 2000 people, must at least 5 have the same BD? Yes, let X be the set consisting of the 2000 people (Pigeons) and Y be the set of 366 days (Pigeon Holes). Define a function $B : X \to Y$ by specifying that $\forall x \in X, B(x) = x$'s birthday. Now $2000 > 4 \times 366 = 1464$ and so by the generalized pigeonhole principle, there must be some birthday $y \in Y$ such that $B^{-1}(y$ has at least $4 + 1 = 5$ elements. Hence at least 5 people must share the same birthday.

Example 4.30

- 12×1967 Penny
- 7×1968 Penny
- 11×1971 Penny

How many must you pick to have at least 5 from the same year? **13**.

9.5 Counting Subsets of a Set: Combinations

Theorem 9.5.1

The number of subsets of size r, called r - combinations, that can be chosen from a set of n elements, $\binom{n}{r}$ is.

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!} \tag{1}$$

Example 9.5.9

$\binom{40}{6} = \frac{40!}{6!34!} = 3,838,380$

Example 9.5.21

How many Morse symbols with 7 or fewer dots or dashes? $2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^7 = 2 \times \frac{2^7-1}{2-1} = 254$

Example 9.6.3

A bakery produces 6 kinds of pastry, including eclairs. Assume a supple of 20 each.

1. How many selections of 20 pastries? *N(T)* = $\binom{20+6-1}{20} = \binom{20}{20} = \frac{25!}{20!5!} =$ $53,130$
2. How many selections of 20 pastries including at least 3 eclairs? $\binom{17+6-1}{17}$ $= \binom{22}{17} = \frac{22!}{17!5!} = 26,334$
3. How many selections of 20 pastries contain at most 2 eclairs? $N(E \leq 2) =$ $N(T) - N(E \geq 3) = 53,130 - 26,334 = 26,796$

Example 9.6.9

```
for k = 1 to n
    for j = k to n
        for i = j to n
```

Number of iterations of inner loops is the same as the number of integers triples (i, j, k) where $1 \leq k \leq j \leq i \leq n$. Triples can be represented as a string of *n - 1* vertical bars and three crosses indicated which integers from *1* to *n* are included.
$\binom{3+n-1}{3} = \binom{n+2}{3} = \frac{n(n+1)(n+3)}{6}$

Example 9.6.12

$y_1 + y_2 + y_3 + y_4 = 30$. How many selections? $\binom{30+4-1}{30}$

Example 9.6.13

$y_1 + y_2 + y_3 + y_4 = 30, y_1 \geq 2, y_2 \geq 2, y_3 \geq 2, y_4 \geq 2.$ $\binom{22+4-1}{22}$

Example 9.6.18

A large pile of coins consist of pennies, nickels, dimes, and quarters.

- How many different collections of 30 coins can be selected? *N(T)* = $\binom{30+4-1}{30)} = \binom{33}{30} =$ 5,456.
- If the pile contains only 15 quarters, but 30 of each other kind of coin, how many collections of 30 can be chosen? $N(Q \geq 16) =$

  Let T be the set of selections of 30 coins, $Q \leq 15$ the set without most 15 quarters, and $Q \geq 6$, The set without at least 16 quarters. Then:

  $T = Q \leq 15 \cup Q \geq 16$
  $Q \leq 15 \cap Q \geq 16 = \emptyset$

$N(T) = N(Q \leq 15) + N(Q \geq 16) - \not N(Q \leq 15 \cap Q \geq 16)$
$N(Q \geq 16) = \binom{14+4-1}{14} = \binom{17}{14} = 680.$
$N(Q \leq 15) = 5,456 - 680 = 4,776$

- 20 Dimes, 30 of each of the rest. How many 30? T $= D \leq 20 \cup D \geq 21$
  Thus: $N(T) = N(D \leq 20) + N(D \geq 21)$
  $N(D \geq 21) = \binom{9+4-1}{9} = \binom{12}{9} = 220$
  $N(Dleq20) = N(T) - N(D \geq 21) = 5,456 - 220 = 5,236$

- 15 dimes, 15 quarters, 30 of each of the rest. How many 30? $N(Q \leq 15 \wedge D \leq 20) = N(T) - N(Q \geq 16 \cup D \geq 21)$
  $= 5,456 - 900 = 4,556$

  $N(Q \geq 16 \wedge D \geq 21) = N(Q \geq 16) + N(D \geq 21) - N(Q \geq 16 \cup D \geq 21)$
  $= 680 + 220 - 0 = 900$

9.1 Graph Theory

9.2 Graph Theory I

10.1.1 Example 10.1.16

A graph of Vertices of degrees 1, 1, 4, 4 and 6 how many edges does the graph have?

$$\begin{aligned} Total\ Edges &= \frac{Add\ Them\ Up}{2} && (1) \\ &= \frac{16}{2} && (2) \\ &= 8 && (3) \end{aligned}$$

10.1.2 Example 10.11g

Draw or explain why doesn't exist: Graph with four vertices of degree 1, 1, 1, and 4.
Doesn't exist, because is of odd total degree.

10.1.3 Example 10.1.20

Draw or explain why doesn't exists: graph with 4 vertices of degrees 1, 2, 3, 4.

10.1.4 Example 10.1.26b

Find all subgraphs of the picture depicted in the book. There's a lot.

10.1.5 Example 10.1.36c

Draw $K_{3,4}$. Three on left, four on right, connect the dots.

10.1.6 Example 10.1.37c

Is the following graph bipartite? If so, redraw.

10.1.7 Example 10.1.3gb

THIS WILL PROBABLY BE ON EXAM.
Find the complement of a graph in the book.

10.1.8 Example 10.1.44

- In a simple graph, must every vertex have degree that is less than the number of vertices in the graph?

  Yes, in a simple graph. This is so because if G is a simple graph with n vertices and v is a vertex of G, then since G has no parallel edges, v can be joined by at most a single edge to each of the other *n - 1* vertices of G and since G has no loops, G cannot be jointed to itself. Thus the max degree of v is *n - 1*.

- Can there be a simple graph that has 4 vertices each of different degree?

  Nah. Suppose there is a simple graph with 4 vertices each of which has different degree. By the previous proof, no vertex can have a greater degree than 3. And, of course, no vertex can have negative degree. Thus, the only possibility is that they have degree 0, 1, 2, and 3. The vertex of degree 3 has to be connected to all 3 other vertices, but this contradicts that one of the other degrees is 0, thus the assumption was false and no such graph exits.

10.2 Trails, Paths, and Circuits

10.2.1 Example 10.2.2

Look up in book.

a) Just walk.

b) Simple Circuit, trail.

c) Closed Walk.

d) Circuit.

e) Trail.

f) Path.

10.3 Matrix Representation of Graphs

Find the adjacency matrix for: *insert picture*
Column x rows, from x to:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

10.3.1 Example 3b

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

10.3.2 Example 6b

Given:

$$\begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Determine without drawing whether this graph is connected.

10.3.3 Example 13

Let 0 denote the matrix:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Find 2×2 matrices A and B such that $A \neq B$, $B \neq 0$, and $AB \neq 0$, but $BA = 0$.

Do this some other time.

10.3 Matrix Representations of Graphs

10.3.1 Example 10.3.19

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 2 & 2 \\ 3 & 2 & 5 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 6 & 3 & 3 \\ 3 & 2 & 2 \\ 3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 15 & 9 & 15 \\ 9 & 5 & 8 \\ 15 & 8 & 8 \end{bmatrix}$$

Assume A is an adjacney matrix. How many walks of length 2 from v_1 to v_3? 3.

Assume A is an adjacney matrix. How many walks of length 3 from v_1 to v_3? 15.

Examine the calculations you performed to find five walks of length 2 form v_3 to v_3. Then draw G and find the walks by visual inspection.

$$2 \cdot 2 + 1 \cdot 1 + 0 \cdot 0 \tag{1}$$

Left edges from v_3 to v_1, right is edges from v_1 to v_3. $2 \times 2 =$ walk of length 2 from v_3 to v_3 via v_1.

10.2 Matrix Representations of Graphs

Graph in book.

- How many from a to c? **4.**
- How many trails from a to c? **4 + 3 + 2 + 1**
- How many walks form a to c? ∞

10.2.1 Example 10.3.6b

An edge whose removal disconnects graph it is part of is called an bridge. Find the bridge in:

The graph is in the book

- $< v_7, v_8 >$
- $< v_3, v_4 >$
- $< v_1, v_2 >$

10.2.2 Example 10.2.8d

Find the number of connected components in: *Image in book.* **2.**

10.2.3 Euler Circuit

Let G be a graph. An Euler circuit for G is a circuit that contains every vertex and every every edge of G. That is, an Euler Circuit for G is a sequence of adjacent vertices and edges in G that has at least one edge, starts and ends in the same vertex, uses every vertex of G at least once, and uses every edge exactly edge exactly one.

10.2.4 Theorem 10.2.2

If a graph has an Euler circuit, then every vertex has even degree.

10.2.5 Theorem 10.2.3

if a graph G is connected and the degree of every vertex is even, then G has an Euler circuit.

10.2.6 Example 10.2.6

Insert graph here.

Test II Study Guide

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5 Sequences, Mathematical Induction, and Recursion

5.1 Sequences

If $a_m, a_{m+1}, a_{m+1} \dots$ and $b_m, b_{m+1}, b_{m+1} \dots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

1. $\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$
2. $c \times \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \times a_k$
3. $(\Pi_{k=m}^{n} a_k) \times (\Pi_{k=m}^{n} b_k) = \Pi_{k=m}^{n} a_k \times b_k$

5.2 Mathematical Induction I

n choose r

For all integers n and r with $0 \leq r \leq n$,

$$\binom{n}{k} = \frac{n!}{r!(n-r)!}$$

Sum of the First n Integers

For all integers $n \geq 1$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Sum of Geometric Sequence

For any real number r except 1, and an integer $n \geq 0$,

$$\sum_{i-0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

6 Set Theory

6.1 Set Theory: Definitions and the Element Method of Proof

Proper Subset

A is a **proper subset** of B $\Leftrightarrow$

1. $A \subseteq B$, and
2. there is at least one element in B that is not in A.

Element Argument: The Basic Method for Proving That One Set Is a Subset of Another

Let sets X and Y be given. To prove that $X \subseteq Y$,

1. **suppose** that x is a particular but arbitrary chosen element of X,
2. **show** that x is an element of Y.

Equals

Given sets A and B, A **equals** B, written $\mathbf{A = B}$, if, and only if, every element of A is in B and every element of B is in A.
Symbolically:

$$A = B \quad \Leftrightarrow \quad A \subseteq B \text{ and } B \subseteq A.$$

Union, Intersection, Difference, Complement

Let A and B be subsets of a universal set U.

1. The **union** of A and B, denoted $A \cup B$, is the set of all elements that are in at least one of A or B.

2. The **intersection** of A and B, denoted $A \cap B$, is the set of all elements that are common to both A and B.

3. The **difference** of B minus A (or **relative complement** of A in B), denoted $B - A$, is the set of all elements that are in B and not A.

4. The **complement** of A, denoted A^c, is the set of all elements in U that are not in A.

Symbolically,

$$\begin{aligned} A \cup B &= \{x \in U \mid x \in A \text{ or } x \in B\}, \\ A \cap B &= \{x \in U \mid x \in A \text{ and } x \in B\}, \\ B - A &= \{x \in U \mid x \in B \text{ and } x \notin A\}, \\ A^c &= \{x \in U \mid x \notin A\}. \end{aligned}$$

Disjoint

Two sets are called **disjoint** if, and only if, they have no elements in common. Symbolically:

$$A \text{ and } B \text{ are disjoint} \quad \Leftrightarrow \quad A \cup B = \emptyset$$

Partition

A finite or infinite collection of nonempty sets $A_1, A_2, A_3 \ldots$ is a **partition** of a set A if, and only if,

1. A is the union of all the A_i

2. The sets $A_1, A_2, A_3, \ldots$ are mutually disjoint. (Not-overlapping)

Power Set

Given a set A, the **power set** of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

Cartesian Product

Gives sets $A_1, A_2, \ldots, A_n$ the **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \ldots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots a_n \in A_n$.
Symbolically:

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2

7 Functions

7.1 Functions Defined on General Sets

Function

A **function *f* from a set *X* to a set *Y***, denoted $f : x \to Y$, is a relation from X, the **domain**, to Y, the **co-domain**, that satisfies two properties: (1) every element X is related to some element in Y, and (2) no element in X is related to more than one element in Y.

Logarithms And Logarithmic Function

Let b be a positive real number with $b \neq 1$. For each positive real number x, the **logarithm with base b of x**, written $\log_b x$, is the exponent to which b must be raised to obtain x. Symbolically,

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x$$

The **logarithmic function with base *b*** is the function from $\mathbf{R}^+$ to $\mathbf{R}$ that takes each positive real number x to $\log_b x$

One-to-One

Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all element x_1 and x_2 in X,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently,

$$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2)$$

Symbolically,

$$F : X \to Y \text{is one-to-one} \quad \Leftrightarrow \quad \forall x_1, x_2 \in X, \text{ if } F(x_1) = F(x_2) \text{ then } x_1 = x_2$$

This can be read as: A function $F : X \to Y$ is *not* one-to-one **if, and only if,** there exist elements $x_1, x_2 \in X$ with $F(x_1) = F(x_2)$ and $x_1 \neq x_2$.

Proof of One-To-One

$f(x) = 4x - 1$
Suppose x_1 and x_2 are real numbers such that $f(x_1) = f(x_2)$.

$$4x_1 - 1 = 4x_2 - 1 \quad (1)$$
$$4x_1 = 4x_2 \quad (2)$$
$$x_1 = x_2 \quad (3)$$
$$(4)$$

Onto

Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that y = *F(x)*.
Symbolically:

$$F : X \to Y \text{ is onto} \quad \Leftrightarrow \quad \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

In other words, $F : X \to Y$is *not* onto **if, and only if,** $\exists y$ in Y such that $\forall x \in X, F(x) \neq y$.

Onto Proof Example

$f(x) = 4x - 1$
Let $y \in \mathbf{R}$. Let $x = (y+1)/4$. Then x is a real number since sums and quotients (other than by 0) of a real numbers are real numbers. It follows:

$$\begin{aligned} f(x) = f(\frac{y+1}{4}) \\ = 4 \times f(\frac{y+1}{4}) - 1 \\ = (y+1) - 1 \\ = y \end{aligned}$$

Bijection

A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function F: X → Y that is *both one-to-one and onto.*

Inverse Image

Suppose *F: X* → *Y* is a one-to-one correspondence; that is, suppose F is one-to-one and onto. Then there is a function $F^{-1} : Y \to X$ that is defined as follows:
Just take the inverse. It's basic algebra.

8 Relations

8.1 Relations on Sets

Relation

Let **R** be a relation from A to B. Define the inverse relation R^{-1} from B to A as follows:

$$R^{-1} = (x, y) \ \in B \times A | (x, y) \in R$$

This is equivelent to: $\forall x \in A$ and $y \in B, (y, x) \ \in R^{-1} \Leftrightarrow (x, y) \in R$

Relation on Sets

A **relation on a set A** is a relation from A to A.

***n*-ary relation**

Given sets $A_1, A_2, \ldots, A_n$, an ***n*-ary relation** R on $A_1 \times A_2 \times \ldots \times A_n$.The special casts of 2-ary, 3-ary, and 4-ary relations are called **binary, ternary,** and **quaternary relations**, respectively.

8.2 Reflexivity, Symmetry, and Transitivity

Reflexivity, Symmetry, and Transitivity

Let R be a relation on a set A.

Reflexive R is reflexive if, and only if, for all $x \in A, x\ R\ x$

- R is reflexive $\Leftrightarrow$ for all x in A, $(x, x) \in R$.
- **Reflexive:** Each element is related to itself.
- R is **not reflexive** $\Leftrightarrow$ there is an element x in A such that $x \not R\ x$ [*that is, such that* $(x, y) \notin R$].

Symmetric R is symmetric if, and only if, for all $x, y \in A$, **if** $x\ R\ y$ then $y\ R\ x$

- R is symmetric $\Leftrightarrow$ for all x and y in A, **if**$(x, y) \in R$ then $(y, x) \in R$
- **Symmetric:** If any one element is related to any other element, then the second element is related to the first.
- R is **not symmetric** $\Leftrightarrow$ there are elements x and y in A such that $x\ R\ y$ but $y\ \not R\ x$ [*that is, such that* $(x, y) \in R$ *but* $(x, y) \notin R$].

Transitive R is transitive if, and only if, for all $x, y, z \in A$, **if** $x\ R\ y$ and $y\ R\ z$ then $x\ R\ z$

- R is transitive $\Leftrightarrow$ for all x, y, and z in A, **if** $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

- **Transitive** If any one element is related to a second and that second element is related to a third, then the first element is related to the third.
- R is **not transitive** $\Leftrightarrow$ there are elements x, y, and z in A such that $x\ R\ y$ and $y\ R\ z$ but $x \not R\ z$ [*that is, such that* $(x, y) \in R$ *and* $(y, z) \in R$ *but* $(x, z) \notin R$]

The Transitive Closure of a Relation

Let A be a set and R a relation on A. the **transitive closure** of R is the relation R^t on A that satisfies the following three properties:

1. R^t is transitive.
2. $R \subset R^t$.
3. If S is any other transitive relation that contains R, then $R^t \subset S$

8.3 Equivalence Relations

Give a partition of a set A, the **relation induced by the partition**, R, is defined on A as follows: For all $x, y \in A$,

$x\ R\ y$ there is a subset A_i of the partition such that both x and y are in A_i

Equivalence Relation

Let A be a set and R a relation on A. R is an **equivalence relation** if, and only if, R is reflexive, symmetric,and transitive.

Equivalence Class

Suppose A is a set and R is an equivalence relation on A. For each element a in A, the **equivalence class of a**, denoted [**a**] and called the **class of a**

Representative

Suppose R is an equivalence relation on a set A and S is an equivalence class of R. A representative of the class S is any element a such that *[a] = S.*

Congruence Modulo

Let m and n be integers and let d be a positive integer. We say that **m is congruent to n modulo d** and write

$$m \equiv n (\text{mod } d)$$

if, and only if,

$$d \mid (m - n)$$

Symbolically:

$$m \equiv n \pmod{d} \Leftrightarrow d \mid (m - n)$$

8.5 Partial Order Relations

Antisymmetric

Let R be a relation on a set A. R is **antisymmetric** if, and only if,

$$\text{for all } a \text{ and } b \text{ in } A, \quad \text{if } a\ R\ b \text{ and } b\ R\ a \text{ then } a = b.$$

Partial Order Relation

Let R be a relation defined on a set A. R is a **partial order relation** if, and only if, R is reflexive, antisymmetric, and transitive.

General Partial Order

Because of the special paradigmatic role played by the $\leq$ relation in the study of partial order relations, the symbol $\preceq$ is often used to refer to a general partial order relation, and the notation $x \preceq y$ is read "x is less than or equal to y" or "y is greater than or equal to x."

Dictionary or Lexicographic

Let A be a set with a partial order relation R, and let S be a set of strings over A. Define a relation $\preceq$ on S as follows:
For any two strings in S, $a_1a_2 \cdots a_m$ and $b_1b_2 \cdots b_n$, where m and n are positive integers,

1. $m \leq n$ $a_i = b_i$ for all $i = 1, 2, \ldots, m$, then

$$a_1 a_2 \cdots a_m \preceq b_1 b_2 \cdots b_n.$$

2. If for some integer k with $k \leq m, k \leq n,$ and $k \geq 1, a_i = b_i$ for all $i = 1, 2, \ldots, k-1$ and $a_k \neq b_k,$ but $a_k \; R \; b_k$ then

$$a_1 a_2 \cdots a_m \preceq b_1 b_2 \cdots b_n.$$

3. If ε is the null string and s in any string in S, then $\epsilon \preceq s$.

If no strings are related other than by these three conditions, then $\preceq$ is a partial order relation.

Maximal, Minimal, Greatest, Least

A *maximal element* in a partially ordered set is an element that is greater than or equal to every element to which it is comparable. (There may be many elements to which it is not comparable.) A *greatest element* in a partially ordered set is an element that is greater than or equal to every element in the set (so it is comparable to every element in the set). Minimal and least are defined similarly.

9.1 Graphs: Definitions and Basic Properties

9.1.1 Graph Terminology

A **graph** G consists of two finite sets: a nonempty set *V(G)* of **vertices** and a set *E(G)* of **edges**, where each edge is associated with a set consisting of either one or two vertices called its **endpoints**. The correspondence from edges to endpoints is called the **edge-endpoint function**.

An edge with just one endpoint is called a **loop**, and two or more distinct edges with the same set of endpoints are said to be **parallel**. An edge is said to **connect** its endpoints; two vertices that are connected by an edge are called **adjacent**; and a vertex that is an endpoint of a loop is said to be **adjacent to itself**.

An edge is said to be **incident on** each of its endpoints, and two edges incident on the same endpoint are called **adjacent**. A vertex on which no edges are incident is called **isolated**.

9.1.2 Directed Graph / Digraph

A **directed graph**, or **digraph**, consists of two finite sets: a nonempty set *V(G)* of vertices and a set *D(G)* of directed edges, where each is associated with an ordered pair of vertices called its **endpoints**. If edge e is associated with the pair (v, w) of vertices, then e is said to be the (**directed**) **edge** from v to w.

9.1.3 Simple Graph

A **simple graph** is a graph that does not have any loops or parallel edges. In a simple graph, an edge with endpoints v and w is denoted v, w.

9.1.4 Complete Graph of n Vertices

Let n be a positive integer. A **complete graph on n vertices**, denoted K_n, is a simple graph with n vertices and exactly one edge connecting each pair of distinct vertices.

9.1.5 Complete Bipartite Graph on (m, n) Vertices

Let m and n be positive integers. A complete bipartite graph on (m, n) vertices, denoted $K_{m,n}$, is a simple graph with distinct vertices $v_1, v_2, \ldots, v_m$ and $w_1, w_2, \ldots, w_n$ that satisfies the following properties: For all $i, k = 1, 2, \ldots, m$ and for all $j, l = 1, 2, \ldots, n$,

1. There is an edge from each vertex v_i to each vertex w_j.
2. There is no edge from any vertex v_i to any other vertex v_k
3. There is no edge from any vertex w_j to any other vertex w_i

9.1.6 Subgraph

A graph H is said to be a **subgraph** of a graph G if, and only if, every vertex in H is also a vertex in G,every edge in H is also an edge in G,and every edge in H has the same endpoints as it has in G.

9.1.7 Degree

Let G be a graph and v a vertex of G. The **degree of *v***, denoted **deg(v)**, equals the number of edges that are incident on v, with an edge that is a loop counted twice. The **total degree of *G*** is the sum of the degrees of all the vertices of G.

9.1.8 The Handshake Theorem

If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges of G. Specifically, if the vertices of G are $v_1, v_2, \ldots, v_n$, where n is a nonnegative integer, then

$$\text{The total degree of } G = deg(v_1) + deg(v_2) + \cdots + deg(v_n) \tag{1}$$

$$= 2 \times (\text{the number of edge of } G) \tag{2}$$

This means that **the total degree of a graph is even**.

9.1.9 Proposition

In any graph there are an even number of vertices of odd degree.

9.1.10 Complement

If G is a simple graph, the **complement of G**, denoted **G'**, is obtained as follows: The vertex set of G is identical to the vertex set of G. However, two distinct vertices v and w of G are connected by an edge if, and only if, v and w are not connected by an edge in G.

9.2 Trails, Paths, and Circuits

9.2.1 Definitions

Let G be a graph, and let v and w be vertices in G.

Walk A **walk from *v* to *w*** finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n \tag{3}$$

where the v's represent vertices, the e's represent edges, $v_0 = v, v_n = w$, and for all $i = 1, 2, \ldots n, v_{i1}$ and v_i are the endpoints of e_i. The **trivial walk from *v* to *v*** consists of the single vertex v.

Trail A **trail** from v to w is a walk from v to w that does not contain a repeated edge.

Path A **path** from v to w is a trail that does not contain a repeated vertex.

Closed Walk A **closed walk** is a walk that starts and ends at the same vertex.

Circuit A **circuit** is a closed walk that contains at least one edge and does not contain a repeated edge.

Simple Circuit A **simple circuit** is a circuit that does not have any other repeated vertex except the first and last.

	Repeated Edge?	**Repeated Vertex?**	**Starts and Ends at Same Point?**	**Must Contain At Least One Edge?**
Walk	allowed	allowed	allowed	no
Trail	no	allowed	allowed	no
Path	no	no	no	no
Closed Walk	allowed	allowed	yes	no
Circuit	no	allowed	yes	yes
Simple Circuit	no	first and last only	allowed	yes

9.2.2 Connected

Let G be a graph. Two **vertices *v* and *w* of G are connected** if, and only if, there is a walk from v to w. The **graph G is connected** if, and only if, given *any* two vertices v and w in G, there is a walk from v to w. Symbolically,

$$G \text{ is connected} \Leftrightarrow \forall \text{ vertices } v, w \in V(G), \exists \text{ a walk from } v \text{ to } w. \tag{4}$$

Let G be a graph.

- If G is connected, then any two distinct vertices of G can be connected by a path.
- . If vertices v and w are part of a circuit in G and one edge is removed from the circuit, then there still exists a trail from v to w in G.
- If G is connected and G contains a circuit, then an edge of the circuit can be removed without disconnecting G.

A graph H is a connected component of a graph G if, and only if,

1. H is subgraph of G;

2. H is connected; and

3. no connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H.

9.2.3 Euler Circuit

Let G be a graph. An **Euler circuit** for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit for G is a sequence of adjacent vertices and edges in G that has at least one edge, starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.

If a graph has an Euler circuit, then every vertex of the graph has positive even degree. If some vertex of a graph has odd degree, then the graph does not have an Euler circuit.

9.2.4 Euler Algorithm

If a graph G is connected and the degree of every vertex of G is a positive even integer, then G has an Euler circuit.
Proof: Suppose that G is any connected graph and suppose that every vertex of G is a positive even integer. *[We must find an Euler circuit for G.]* Construct a circuit C by the following algorithm:

Step 1: Pick any vertex v of G at which to start. *[This step can be accomplished because the vertex set of G is nonempty by assumption.]*

Step 2: Pick any sequence of adjacent vertices and edges, starting and ending at v and never repeating an edge. Call the resulting circuit C. *[This step can be performed for the following reasons: Since the degree of each vertex of G is a positive even integer, as each vertex of G is entered by traveling on one edge, either the vertex is v itself and there is no other unused edge adja- cent to v, or the vertex can be exited by traveling on another previously unused edge. Since the number of edges of the graph is finite (by definition of graph), the sequence of distinct edges cannot go on forever. The sequence can eventu- ally return to v because the degree of v is a positive even integer, and so if an edge connects v to another vertex, there must be a different edge that connects back to v.]*

Step 3: Check whether C contains every edge and vertex of G. If so, C is an Euler circuit, and we are finished. If not, perform the following steps.

Step 3a: Remove all edges of C from G and also any vertices that become isolated when the edges of C are removed. Call the resulting subgraph G . *[Note that G may not be connected (as illustrated in Figure 10.2.4), but every vertex of G has positive, even degree (since removing the edges of C removes an even number of edges from each vertex, the difference of two even integers is even, and isolated vertices with degree 0 were removed.)]*

Step 3b: Pick any vertex w common to both C and G . *[There must be at least one such vertex since G is connected. (See exercise 44.) (In Figure 10.2.4 there are two such vertices: u and w.)]*

Step 3c: Pick any sequence of adjacent vertices and edges of G , starting and ending at w and never repeating an edge. Call the resulting circuit C . *[This can be done since each vertex of G has positive, even degree and G is finite. See the justification for step 2.]*

Step 3d: Patch C and C together to create a new circuit C as follows: Start at v and follow C all the way to w. Then follow C all the way back to w. After that, continue along the untraveled portion of C to return to v. *[The effect of executing steps 3c and 3d for the graph of Figure 10.2.4 is shown in Figure 10.2.5.]*

Step 3e: Let $C = C$ and go back to step 3.

Since the graph G is finite, execution of the steps outlined in this algorithm must eventually terminate. At that point an Euler circuit for G will have been constructed. (Note that because of the element of choice in steps 1, 2, 3b, and 3c, a variety of different Euler circuits can be produced by using this algorithm.)

9.2.5 Theorem

A graph G has an Euler circuit if, and only if, G is connected and every vertex of G has positive even degree.

9.2.6 Corollary

Let G be a graph, and let v and w be two distinct vertices of G. There is an Euler path from v to w if, and only if, G is connected, v and w have odd degree, and all other vertices of G have positive even degree.

9.2.7 Hamiltonian Circuit

Given a graph G, a Hamiltonian circuit for G is a simple circuit that includes every vertex of G. That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, except for the first and the last, which are the same.

If a graph G has a Hamiltonian circuit, then G has a subgraph H with the following properties:

1. H contains every vertex of G.
2. H is connected.
3. H has the same number of edges as vertices.
4. Every vertex of H has degree 2.

10.3 Matrix Representations of Graphs

An **m** $\times$ **n** (read "m by n") **matrix A over a set S** is a rectangular array of elements of S arranged into m rows and n columns.

We write $\mathbf{A} = (a_{ij})$

10.3.1 Adjacency Matrix

Let G be a directed graph with ordered vertices $v_1, v_2, \ldots, v_n$. The adjacency matrix of G is the $n \times n$ matrix $\mathbf{A} = (a_{ij})$ over the set of nonnegative integers such that

$$a_{ij} = \text{the number of arrows from } v_i \text{ to } v_j \text{ for all } i, j = 1, 2, \ldots, n. \quad (1)$$

10.3.2 Adjacency Matrix

Let G be an undirected graph with ordered vertices $v_1, v_2, \ldots, v_n$. The adjacency matrix of G is the $n \times n$ matrix $A = (a_{ij})$ over the set of nonnegative integers such that

$$a_{ij} = \text{the number of edges connecting } v_i \text{ and } v_j \quad (2)$$

for all $i, j = 1, 2, \ldots, n$.

10.3.3 Symmetric

An $n \times n$ square matrix $A = (a_{ij})$ is called **symmetric** if, and only if, for all $i, j = 1, 2, \ldots, n$,

$$a_{ij} = a_{ji} \quad (3)$$

10.3.4 Identity Matrix

For each positive integer n, the $n \times n$ **identity matrix**, denoted $I_n = (\delta ij)$ or just **I** (if the size of the matrix is obvious from context), is the $n \times n$ matrix in which all the entries in the main diagonal are 1's and all other entries are 0's. In other words,

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \quad \text{for all } i, j = 1, 2, \ldots, n$$

,

10.3.5 The Number of Walks

If G is a graph with vertices $v_1, v_2, \ldots, v_m$ and A is the adjacency matrix of G, then for each positive integer n and for all integers $i, j = 1, 2, ldots, m$,

$$\text{the } ij\text{th entry of } A_n = \text{the number of walks of length } n \text{ from } v_i \text{ to } v_j.$$

10.5 Trees

10.5.1 Terminology

A graph is said to be **circuit-free** if, and only if, it has no circuits. A graph is called a **tree** if, and only if, it is circuit-free and connected. A **trivial tree** is a graph that consists of a single vertex. A graph is called a **forest** if, and only if, it is circuit-free and not connected.

10.5.2 Vertex-Degree Relation

Any tree that has more than one vertex has at least one vertex of degree 1.

10.5.3 Terminal Vertex

Let T be a tree. If T has only one or two vertices, then each is called a **terminal vertex**. If T has at least three vertices, then a vertex of degree 1 in T is called a **terminal vertex** (or a **leaf**), and a vertex of degree greater than 1 in T is called an **internal vertex** (or a **branch vertex**).

10.5.4 Edges And Vertices

For any positive integer n, any tree with n vertices has n - *1* edges.

10.6 Connected, Circuit

If G is any connected graph, C is any circuit in G, and any one of the edges of C is removed from G, then the graph that remains is connected.

CS1510

Data Structures

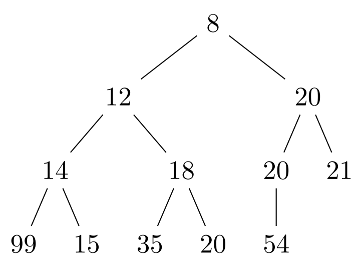
1 Graph

Is there a path from root a to node b? Let's algorithmize!

```
// Recursive
pathSearch(Graph G, start, goal, visited nodes) {
    if start == goal { return true; }

    add started to visited nodes
    for every neighbor x of start not in visited nodes
        solved = pathSearch(G, x, goal, visited nodes)
        if solved {
            return true
        }

    return false;
}

// Not Recursive
pathSearch(Graph G, srat, goal, Visited) {
    stack of nodes S
    push(S, start)

    while (S is not empty) {
        X = top(S)
        pop(S);

        if (x == goal)
            return true

        add x to visited

        for every neighbor y of x, not in viisted
            push (S, y)
    }

    return false;
}
```

Heap

- Binary Search Tree
- All but the bottom level is complete.
- From every node x, x is less than it's two children.
- Member functions.
  - top(T) = *8*
  - insert(T, x)
  - remove(T) = $T =$ *the top of the heap*
- Maintenance Functions
  - Percolate Up, during insertion.
    - Let the item bubble up.
  - Percolate Down, during removal.
    - Like a stone sinking through a viscous liquid.



D.S. ArrayHeap

```
class ArrayHeap {
    T *data;
    int m_max, m_size;

public:
    const T& top() {
        if (m_size != 0) {
            return m_data[0];
        } else {
            cerr << "Shit."
        }
    }

    void insert(const T& x) {
```

```
        if (m_max == m_size) {
            grow();
        }

        int hole = m_size;
        m_size++;

        while (hole > 0 && x < m_data[(hole- 1) / 2]) {
            m_data[hole] = m_data[(hole - 1) / 2];
            hole = (hole - 1)/2;
        }

        m_data[hole] = x;
    }

    void remove() {
        if (m_size == 0) { return; }
        int hole = 0;
        m_size--;

        We can continue this over break.
    }
};
```

- Homework 5 is on stacks
- Go to megaminer.

1 Recursive Object

An object which potentially consists or defined in terms of itself. Recursion is used to define things such as:

- Sets
- Functions
- Other objects

Power of Recursion:

- Describe an infinite object
- Through finite means.

Recursive Definition

- Base Case
- Recursive Case

Example: Set of all strings of balanced parenthesis.
Base Case: () is in the set.
Recursive Case: if s is in the set, then s(), ()s, and (s) are also in the set.

$$Fibonacci = \begin{cases} fib(1) = 1 \\ fib(2) = 1 \\ fib(n) = fib(n-1) + fib(n-2) \end{cases} \quad (1)$$

$$Factorial = \begin{cases} 1! = 1 \\ n! = n \times (n-1) \end{cases} \quad (2)$$

1.1 Recursive Algorithms

- Base Case
  - Direct solution to a small problem instance.
- Recursive Case
  - Decompose problem into smaller instances.
  - Solve smaller instances.
  - Construct solution from smaller solutions.

1.1.1 Triomino Problem

Suppose we have four possible tiles made of three squares.

Problem: Cover $2^n \times 2^n$ board, where one tiles is a hole with triominoes.

- $n = 4, 2^n = 16$

  Morales gives example.

- split board in 4 equal parts.
- Place triomino across 3 split parts without a hole.
- Solve each subpart.

```
void foo() {
  int x;

  foo();
}

quicksort(array, left, right) { // assuming left < or = right
  if (left = right) {
    return; // Base Case
  }

  pivot = a[(left + right) / 2];
```

```
  int i = left;
  int j = right;

  repeat
    while (a[i] < pivot) { i++; }
    while (a[j] > pivot) { j--; }
    if (i < j) {
      swap(a[i], a[j]);
      i++;
      j--;
    }
  while (j > i);

  quicksort(a, i + 1, r);
  quicksort(a, l, i - 1)

}
```

2 Recursive Backtracking

```
try
    intialize choices
    do
        select choice
        if choice is valid
            record choice
            if solution complete
                success!
            else
                try next step
                if next step succeeds
                    success!
                else
                    cancel record
    while !success & more choices available

path_find(grid, int row, int col) {
```

```
    for choice c in {N, NE, E}
        nrow = row after c;
        ncol = col after c;

        if (grid[nrow][ncolumn]) != obstacle && nrow, ncol is in bounds)
        record nrow, ncol;

        if (grid[row][column] == cake!) {
            return true;
        } else {
            solve = path_find(grid, nrow, ncol)
            if (solve) {
                return true;
            } else {
                record C;
            }
        }

    return false
}

bool valid(grid, int r, int c) {
    if (c < 0 || r >= N) {
        return false;
    }
    if (c < 0 || c >= N) {
        return false;
    }
    if (grid[r, c] == obstacle) {
        return false;
    }

    return true;
}

for (int i = 0; i < 3; c++) {
    nrow = col + dir[c][0];
    ncol = col + dir[c][1];
```

```
    if (valid(grid, nrow, ncol, n)
}
```

1 Tree

- A tree is a collection of elements with a hierarchical relation over such elements.
- The actual definition is as follows: The empty collection is a tree (the empty tree). A single element is usually called a node. Node is a tree.
- If n is a node and $T_1, T_2, T_3, \ldots, T_n$ are trees, then n related to $T_1, T_2, \ldots, T_n$:
  - n is called the root
  - $T_1 \ldots T_n$ are called the subtrees of n.
- A *path* is a sequence of nodes. $< a_1, a_2, a_3, \ldots, a_n >$ when n_{i+1} is a parent of $n_i 0 \leq i < n$.
- The *depth* of a node a is the number of nodes in the path from a to the root.
- the *height* of a tree is the greatest depth of a node in the tree.

Note Every tree has only one root.

Child The root of each subtree T_1, T_n are called the children of n

n Is called the parent of the root of each subtree T_1, T_n

Siblings If two roots have the same parent they are called siblings.

Leaf A leaf is a node with no children.

Degree The degree of a node a is the number of children of a.

- The degree of a tree is the highest degree of a node in the tree.

decedent/ancestor If there is a path from node a to node z then a is called a decedent of z. z is called an ancestor of a. The root is every node's ancestor.

1.1 Binary Search Tree

- Binary: Degree 2
- Search Conditions
  - find(T,x)
  - getMin(T)
  - getMax(T)
  - insert(T)
  - remove(T,x)

```
// Data Structure BinaryTree

template <classname T>
class TreeNode {
    T m_data;
    TreeNode *m_right;
    TreeNode *m_left;
};

// To use recursion, functions cannot be a method of TreeNode
const T& getMin(TreeNode *t) {
    if (t == nullptr) { /* error */ }

    if (t -> m_left == nullptr) {
        return t-> m_data;
    } else {
        return getMin(t -> m_left);
    }
}

const T& getMax(TreeNode *t) {
    if (t == nullptr) { /* error */ }
    TreeNode *p = t;

    while (p -> m_right != nullptr) {
        p = p -> m_right;
    }

    return p -> m_data;
}

bool T& find(TreeNode *t, const T& x) {
    if (t == nullptr) { return false; }
    if (t -> m_data == x) { return true; }

    if (x < t -> m_data) {
        return find(t - > m_left, x);
    } else if (x > t -> m_right) {
        return find(t -> right, x);
    }
}

void insert(TreeNode * &t, const T& x) {
    if (t == nullptr) {
        t = new TreeNode;
        t -> m_right = nullptr;
```

```
        t -> m_left = nullptr;
    } else (x < t -> m_data) {
        insert(t -> m_left);
    } else if (x > t -> m_data) {
        insert(t -> m_right);
    } else {
        return; // This is a duplicate. No duplicates allowed.
    }
}

void remove(TreeNode * &t, const T& x) {
/*
3 Cases:
- No Children
- One Child
- Two Children

To remove, you have choice. Max of Left or Min of right.
*/

    if (t == nullptr) {
        return;
    }
    if (x < t -> m_data) {
        remove(t -> left, x);
    } else if (x > t -> m_data) {
        remove(t -> right x);
    } else {
        // FOUND X!
        if (t -> m_right == nullptr && t -> m_left == nullptr) {
            // No children
            delete t;
            t = nullptr;
        } else if (t -> m_right == nullptr ||
                   t-> m_left == nullptr) {
            TreeNode *temporary = t -> m_right;
            if (temporary == nullptr) {
                temporary = t -> left;
            }

            // Now, temporary points to the chlid
            delete x;
            t = temporary;
        } else {
            // X has two children
            t -> m_data = getMin(t -> m_right);
```

```
            remove(t -> m_right, t -> m_data);
        }
    }
}
```

- collection of objects
- repetition is not allows
  - SETS!
- Why? Who cares? WHY NOT VECTORS?
  - find()
    * Find of size 500. $log_2 500 = 8.9$
    * 5000. $log_2 5000 = 12.2$
    * 5 Million. $log_2 5 Million = 22.25$
    * Most important operations.
  - insert()
    * $log_2 n$
  - remove()
    * $log_2 n$

index at i, right(i) $= 2i + 1$, left(i) $= 2i + 2$. The parent of is $\frac{i-2}{2}$

CS2300

Databases

1 Categories of Data Models

- Conceptual data model
- Implementation data model
- Physical data model

2 Three Schema Architecture

Internal Schema Physical data model.

- Physical storage structure
- Data storage details, access path

Conceptual Schema Conceptual or implementation data model.

- Structure of the entire database
- Entities, data types, relationships

External Schema Describe the various user views.

- Usually uses the same data model as the conceptual schema.

3 Data Independence

3.1 Logical Data Independence

The capacity to change the conceptual schema without having to change the external schemas and their associated application programs.

3.2 Physical Data Independence

The capacity to change the internal schema without having to change the conceptual schema.

1 Itinerary

1. Introduction
2. Schedules
3. Ideas
    - Store Front
    - Publishing Medium
    - Social Network
4. Expectations
    - Good Grade?
    - Minimal Effort?
    - Learning?
5. Technology
    - Client Side (Front End)
        - HTML5/CSS3
        - JavaScript
            * Ajax
            * React
            * Angular.js
            * CoffeeScript
            * Ember.js
            * Backbone.js
            * Knockout.js
    - Server Side (Back End)
        - Node.JS
        - Google Go
        - Scala
        - Clojure
        - Rust
        - Perl
        - Ruby
        - PHP
        - C++
        - C#
    - Database
        - SQL
6. Wrap Up

2 Notes

- Everyone is in the Google Calendar, just add your schedule so we know when to meet.

End of Semester Material

Illya Starikov

March 24, 2016

Because it is the end of the semester and we have gotten off track with the assignment, I have decided to clarify expectations and rework some things.

1 Changelog

The follow noticeable changes have occurred:

Dropbox Is no longer going to suffice. Everything is moved to `git`.

- Contact me if there is any issue with the transition.

Subsystems Are now going to be split different. They are listed below.

Front-End Development *Jason Young and Illya Starikov.* This will most likely 2/3 to 1/3, being heavy on Jason.

Back-End Development *Jason Young.* I am very unfamiliar with Django so it will be up to Jason for the extensive part of the backend development.

Database Development *Claire Trebing and Illya Starikov.* The work load will be mostly be a 1/2 to 1/2.

2 Meetings

Meetings have been wishy-washy the last few months, but as we are approaching the end of the semester we are going to need to more frequently meet to discuss how the project is going. **Until the your subsystem is finished, meetings will occur.**

3 Project Phase III

Project Phase III has three parts:

Source Code This will be covered by the Database subsystem.

Report This will be covered by Claire and I.

- The revisions will be made by me.
- The user manual will be made by Claire.

For the user manual, the deliverable is a PDF, Word Document or Google Document. It **must be able to render in plain text** (i.e. I have to be able to paste it into `Sublime Text` without any formatting issues). Please account for this. The document will be gutted and reformatted to LaTeX.

The manual must be broken down into sections, subsection, and subsection or subsections (or, subsubsection). Please indicate these in some manner (whether it be verbosely written or through formatting). This is used for a table of contents, to keep in mind.

Formatting such as *italicizing*, **boldfacing** and the like are encouraged. I will take care of make sure they render correctly. Equations of any form (from $\sum \vec{F} = m\vec{a}$ to $\frac{\hbar^2}{2m}\nabla^2\Psi + V(\mathbf{r})\Psi = -i\hbar\frac{\partial\Psi}{\partial t}$) are encouraged but must be easily readable. Any tables, figures, images and the like must be coherent and easy to interpret.

4 Homework #3 & #4

The last two homeworks are SQL-based, so for the last two homeworks:

- Everyone will finish their own homework.
  - Upon completion, email to me.
  - I will run it through a `diff` utility.
    - Any noticeable disparities will be discussed with respective authors.
- When everyone's homework matches, *Claire* will submit the SQL file.
- Have the homework ready at least two days in advance.

CS2500

Algorithms

1 Algorithms Introduction

- Insertion Sort Example

```
Input: 503 87 512 61 908 170 897

/* The brackets signify the sorted array */

[503] 87 512 61 908 170 897
[87 503] 512 61 908 170 897
[87 503 512] 61 908 170 897
[61 87 503 512] 908 170 897
[61 87 503 512 908] 170 897
[61 87 170 503 512 908] 897
[61 87 170 503 512 908] 897
[61 87 170 503 512 897 908]
```

- Do the take home examples.

2 Asymptotic Analysis

- Big-O

$$O(g(n)) = \{f(n) : \text{there exists positive constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le cg(n) \,\forall\, n \ge n_0\} \tag{1}$$

2 Asymptotic Analysis

2.1 Proofs

Prove the following for big O.

$$
\begin{aligned}
n^2 + 2n + \lg n = O(n^2) && (1)\\
\leq n^2 + 2n + \lg n && (2)\\
\leq n^2 + 2n^2 + \lg n \cdot n && (3)\\
\leq n^2 + 2n^2 + n^2 && (4)\\
= 4n^2 && (5)
\end{aligned}
$$

So $c = 4$ and $n_0 = 1$.

2.2 Proof II

Prove the following for big Ω

$$
\begin{aligned}
2n^3 - 3n^2 + 2n = \Omega(n^3) && (6)\\
= 2n^3 - 3n^2 + 2n && (7)\\
\geq 2n^3 - 3n^2 && (8)\\
\geq 2n^3 - n^3, \quad n \geq 3 && (9)\\
= n^3 && (10)
\end{aligned}
$$

So $c = 1$, $n_0 = 3$.

2.3 Proof III

Upper bound.

$$
\begin{aligned}
\frac{n^2}{2} - \frac{n}{2} \quad &= \quad \Theta(n^2) && (11)\\
&= \quad \frac{n^2}{2} - \frac{n}{2} && (12)\\
&\leq \quad \frac{n^2}{2} - \frac{n}{2} && (13)\\
&= \quad \leq \frac{n^2}{2}, c_2 = \frac{1}{2}, n \geq 0. && (14)
\end{aligned}
$$

Lower bound.

$$
\begin{aligned}
&\quad \frac{n^2}{2} - \frac{n}{2} && (15)\\
&\geq \quad \frac{n^2}{2} - \frac{n}{2} * \frac{n}{2} && (16)\\
&= \quad \frac{1}{4}n^2 && (17)
\end{aligned}
$$

$c_1 = \frac{1}{4}$ and $n_0 = 2$

2.4 Growth Classes Proof

Show $n^3 - 10n^2 \neq O(n^2)$

$$
\begin{aligned}
0 &\leq n^3 - 10n^2 \leq cn^2 && (18)\\
n^3 &\leq 10n^2 + cn^2 && (19)\\
n^3 &\leq (10+c)n^2 && (20)\\
n &\leq 10 + C && (21)
\end{aligned}
$$

This is a contradiction.

2.5 Growth Classes Example II

Show $2^n = o(3^n)$

$$
\begin{aligned}
2^n &= o(3^n) && (22)\\
lim_{n\to\infty}\frac{2^n}{3^n} &= lim_{n\to\infty}\frac{2}{3}^n && (23)\\
&= 0 && (24)
\end{aligned}
$$

3 Divide and Conquer

Search for 64.

```
10 20 37 43 52 76 85 93
10 20 37 *43* | 52 76 85 93 [43 vs 64]
52 *76* | 85 93 [76 vs 64]
*52* | 76 [52 vs 64]
```

Three comparisons.

3.1 Merge Sort

```
38 27 43 3 | 9 82 10
38 27 | 43 3       9 82 | 10
38 | 27    43 | 3       9 | 82     10
38        27    43       3       9         82     10
27 38      3 43          9 82       10
3 27 28 43      9 10 82
3 9 10 38 43 82
```

3.2 Substitution Method

$2^n - 1$
Base case: $2 - 1 = 1$
Inductive:

$$
\begin{aligned}
& && (1)\\
T(n) &= 2^n - 1 && (2)\\
T(n+1) &= 2^{n+1} - 1 && (3)\\
&= 2\ T(n) + 1 && (4)\\
&= 2(2^n - 1) && (5)\\
&= 2^{n+1} - 1 && (6)
\end{aligned}
$$

4 Recurrence

Substitution Example

T(n) = cn
T(n+1) = c(n + 1)
= T(n) + c
= cn + c
= c(n + 1)

Recursion Tree

$2T(n/2) + n^2$.

Master Theorem

Check given $T(n)$ is the form of $aT(n/b) + f(n)$

$$T(n) = 8T(n/2) + n^2 \tag{1}$$

$a = 8, b = 2, f(n) = n^2$. Prove $f(n) \in O(n^{log_2 a - \epsilon}) = O(n^{3-\epsilon}) = O(n^{3-1}) = O(n^2)$. $T(n) = \Theta(n^{log_b a}) = \Theta(n^3)$

4 Recurrences

$$
\begin{aligned}
T(n) &= 2T(n/2) + n^3 && (1)\\
\log_b a &= \log_2 2 = 1 && (2)\\
log^{b^a} &= n && (3)\\
n^3 & \in \Omega(n^{\log_b a + 2}) && (4)\\
& \Omega(n^3) && (5)\\
af(n/b) &\leq cf(n) && (6)\\
2f(n/2) &\leq cf(n) && (7)\\
2 \times (\frac{n}{2})^3 &\leq cn^3 && (8)\\
\frac{n^3}{4} &\leq cn^3 && (9)\\
& \text{choose } c < 1 && (10)\\
& \text{choose } c = \frac{1}{4} && (11)\\
& T(n) \in \Theta(n^3) && (12)
\end{aligned}
$$

5 Correctness

Loop has three parts, init, maintenance, termination.

6 Quicksort

6 Quick Sort

1. wall, current, pivot

    - Array ≤ pivot ≤ Array

```
let wall = |, pivot = []
| 2 9 5 6 10 0 7 [4]
  2 | 9 5 6 10 0 7
    2 0 | 5 6 10 9 [7]
      0 2 -> 5 -> 6 | 10 9
        5 6 | 10 9

Output: 0 2 5 6 7 9 10
```

7 Heapsort

```
10 9 7 8 4 6 1 3 5 2 -> 10 | 9 7 | 8 4 6 1 | 3 5 1 0 0 0 0 0 |
15 5 11 1 4 10 7 3 9 -> 15 | 5 11 | 1 4 10 7 | 3 9 0 0 0 0 0 |
                    9    15 | 9 11 | 5 4 10 7 | 3 1 0 0 0 0 0 |
                    1    15 | 5 11 | 1 4 10 7 | 3 9 0 0 0 0 0 |
                    3    15 | 5 11 | 1 4 10 7 | 3 9 0 0 0 0 0 |
                    =    15 | 5 11 | 1 4 10 7 | 3 9 0 0 0 0 0 |
```

The follow the while loop for the heapsort.

8 Counting Sort Algorithm

- We have talked about many algorithms so far

  Insertion Sort Worst: $\Theta(n^2)$, Average: $\Theta(n^2)$

  Merge Sort Worst: $\Theta(n \lg n)$, Average: $\Theta(n \lg n)$

  Quick Sort Worst: $\Theta(n^2)$, Average: $\Theta(n \lg n)$

  Heap Sort Worst: $\Theta(n \lg n)$, Average: $\Theta(n \lg n)$

- Most of the basic operations have been so far is comparison.
- Merge sort is asymptotically optimal (in general).
- Any comparison based sorting is bounded $\Omega(n \lg n)$
- Is there any way to sort in linear time?
  - Counting Sort is a way to achieve this.

8.1 Counting Sort

We want to sort integers. $[0 \ldots k]$, where k is max in the array.

1. Count the number of each distinct elements (frequency).
2. Determine the position of each element by placing it to the "counting index".

Specifically,

1. Specifically, given array of size n, and the maximum integer in the array is k.
2. Create "counting array", whose size is $k + 1$.
3. Fill the counting array with the frequency of each element.
4. Update the counting array accumulatively adding up frequencies
5. Place our numbers in the position of "accumulated numbers".

8.1.1 Example

```
Given Array: [1 3 0 2 4 3 1]
Size: 7
Max: 4

Counting array (size of k+1): [1 | 2 | 1 | 2 | 1]
Notice how the indexes are how how many times the number appears in original array
```

```
Update: [1 3 4 6 7]
These are the updated positions of the array elements

Enumerated Steps
Output: [ | | 1 | | | |]
Now deincrement updated array. ([1 2 4 6 7])

Output: [ | | 1 | | | 3 |] ([1 2 4 5 7])
Output: [ | | 1 | | | 3 | 4] ([1 2 4 5 6])
So on and so forth.
```

8.1.2 Psuedocode

Let A = input array, B = output array, n = size of array, k = maximum number.

```
countingSort(A, B, n, k)
    let C[0...k] be the counting array
    for i = 0 to k
        C[i] = 0
    for j = 1 to n
        C[A[j]] = C[A[j]] + 1
    for i = 1 to k
        C[i] = C[i] + C[i - 1]
    for j = n down to 1
        B[C[A[j]]] = A[j]
        C[A[j]] = C[A[j]] - 1
```

We can see that the complexity is $\Theta(n + k)$

9 Examples

9.1 Counting Sort Algorithm

```
4 6 4 1 3 4 1 4

Assume [0...k]. Max integer: 6, size of array: 8.

Original Array: [ 4 | 6 | 4 | 1 | 3 | 4 | 1 | 4 ]
Counting Array: [ 0 | 2 | 0 | 2 | 3 | 0 | 1 ] (size is max + 1)
Update Array:   [ 0 | 2 | 2 | 4 | 7 | 7 | 8 ]
Output Array:   [ 1 | 1 | 3 | 4 | 4 | 4 | 6 ]
(reverse order of original array via indexes, the output is not index based)
```

9.2 Growth Classes Example

Show $20n^2 + 2n + 5 = O(n^2)$

$$O(n^2) = 20n^2 + 2n + 5 \quad (1)$$
$$= 20n^2 + 2n^2 + 5n^2 \quad (2)$$
$$= 27n^2 \quad (3)$$

So $n_0 = 1$ and $C = 27$.

9.3 Growth Example II

$5n + 7 = o(n^2)$. For every c, find $n_0 > 0$

$$5n + 7 \leq 5n + n = 6n < cn^2 \quad (4)$$

For $n_0 = 7$, this holds. But now we have to prove $\forall c$. We do so via limit.

9.4 Growth Class Example

Prove $5n^2 + 2n = \Theta(n^2)$. Just take big-O and big-Ω

9.5 Growth Class Example

Prove $5n^2 15n + 100 = \Omega(n^2)$

$$> 5n^2 - 15n \quad (5)$$
$$\geq 5n^2 - n^2 \quad (6)$$
$$= 4n^2 \quad (7)$$

9.6 Master Theorem

10 Dynamic Programming

```
F(2) = F(0) + F(1) = 1
F(3) = F(2) + F(1) = 2
```

11 LCS

	1	2	3	4
1	0	36	84	124
2	X	0	72	132
3	X	X	0	120
4	X	X	X	0

So $((A_1 A_2) A_3) A_4$

Test I Study Guide

Illya Starikov

February 15, 2016

1 Sorting Algorithms

1.1 Insertion-Sort

```
Insertion-Sort(A)
    for j = 2 to A.length
        key = A[j]
        // Insert A[j] into the sorted sequence A[1..j - 1]
        i = j - 1
        while i > 0 and A[j] > key
            A[i + 1] =  A[i]
            i = i - 1
        A[i + 1] = key
```

1.2 Merge Sort

```
Merge-Sort(A, p, r)
    if p < r
        q = ⌊(q + r) /2⌋
        Merge-Sort(A, p, q)
        Merge-Sort(A, q + 1, r)
        Merge(A, p, q, r)

Merge(A, p, q, r)
    n_1 = q - p + 1
    n_2 = r - q
    let L[1...n_1 + 1] and R[1..n_2] be new arrays
    for i = 1 to n_1
        L[i] = A[p + i - 1]
    for j = 1 to n_2
        R[j] = A[q + j]
    R[n_1 + 1] = ∞
    R[n_2 + 1] = ∞
    i = 1
    j = 1
```

```
    for k = p to r
        if L[i] ≤ R[j]
            A[k] = L[i]
            i = i + 1
        else A[k] = R[j]
            j = j + 1
```

1.3 Heap Sort

```
Max-Heapify(A, i)
    l = Left(i)
    r = Right(i)
    if l ≤ A.heap-size and A[l] > A[i]
        largest = l
    else largest = i
    if r ≤ A.heap-size and A[r] > A[largest]
        largest = r
    if largest ≠ i
        exchange A[i] with A[largest]
        Max-Heaphify(A-largest)

Heapsort(A)
    Build-Max-Heap(A)
    for i = A.length downto 2
        exchange A[1] with A[i]
        A.heap-size = A.heap-size - 1
        Max-Heapify(A, 1)
```

1.4 Quicksort

```
Quicksort(A, p, r)
    if p < r
        q = Partition(A, p, r)
        Quicksort(A, p, q - 1)
        Quicksort(A, q + 1, r)

Partition(A, p, r)
    x = A[r]
    i = p - 1
    for j = p to r - 1
        if A[j] ≤ x
            i = i + 1
            exchange A[i] with A[j]
    exchange A[i + 1] with A[r]
    return i + 1
```

1.5 Counting Sort

Let A = input array, B = output array, n = size of array, k = maximum number.

```
countingSort(A, B, n, k)
    let C[0...k] be the counting array
    for i = 0 to k
        C[i] = 0
    for j = 1 to n
        C[A[j]] = C[A[j]] + 1
    for i = 1 to k
        C[i] = C[i] + C[i - 1]
    for j = n down to 1
        B[C[A[j]]] = A[j]
        C[A[j]] = C[A[j]] - 1
```

1.6 Rate of Growth

Insertion Sort Worst: $\Theta(n^2)$, Average: $\Theta(n^2)$

Merge Sort Worst: $\Theta(n \lg n)$, Average: $\Theta(n \lg n)$

Quick Sort Worst: $\Theta(n^2)$, Average: $\Theta(n \lg n)$

Heap Sort Worst: $\Theta(n \lg n)$, Average: $\Theta(n \lg n)$

2 Growth Classes

2.1 O-notation

$O(g(n)) = \{f(n) :$ there exists positive constant c and n_0 such that $0 \leq f(n) \leq c\, g(n)$ for all $n \geq n_0\}$

2.2 Ω-notation

$\Omega(g(n)) = \{f(n) :$ there exists positive constants c and n_0 such that $0 \leq c\, g(n) \leq f(n)$ for all $n \geq n_0\}$

2.3 Θ-notation

$$\Theta(g(n)) = \{f(n) : \text{ there exists positive constants } c_1, c_2 \text{ and } n_0 \text{ such that } 0 \leq c_1\ g(n) \leq f(n) \leq c_2\ g(n) \text{ for all } n \geq n_0\}$$

2.4 o-notation (Little-o)

$f(n) \in o(g(n))$ iff for every $c > 0$ there is an $n_0 > 0$ such that

$$0 \leq f(n) < c\ g(n)$$

for all $n \geq n_0$

2.5 ω-notation (Little-omega)

$$f(n) \in \omega(g(n)) \text{ iff } g(n) \in o(f(n)).$$

or $f(n) \in \omega(g(n))$ iff for every $c > 0$ there is an $n_0 > 0$ such that

$$0 \leq c\ g(n) < f(n)$$

for all $n \geq n_0$

3 Master Theorem

The master theorem can only be applied to recurrence equations of the form:

$$T(n) = aT(n/b) + f(n)$$

3.1 Constants

n The size of the problem

a The number of subproblems

n/b The size of each subproblem

f(n) cost outside of recursive calls (divide, combine)

3.2 Cases

$f(n) \in O(n^{\log_b a-\epsilon})$	$T(n) \in \Theta(n^{\log_b a})$
$f(n) \in \Theta(n^{\log_b a})$	$T(n) \in \Theta(n^{\log_b a} \lg n)$
$f(n) \in \Omega(n^{\log_b a+\epsilon})$	$T(n) \in \Theta(f(n))$

4 Definitions

Algorithm Any well-defined computational procedure that takes a set of values as input and produces a set of values as output in a finite number of steps

Correct Algorithm One returns the correct solution for every valid instance of a problem

Loop Invariance Define a key property about the relationship among variables of the algorithms.

- Holds in the *initial* case.

  Initialization The loop invariance must be true prior to the first iteration.

- Is *maintained* each iteration.

Maintenance If the property holds prior to an iteration, it must still hold after the iteration is complete.

- Yields correctness when the loop *terminates*.

  Termination The invariant provides a useful property that helps demonstrate the algorithm is correct.

Heapify Go all the way down to the heap and fix the violations of the max-heap property by sifting-up

- Dynamic Programming Two Properties

  Overlapping Subproblems A recursive solution contains a small number of distinct subproblems repeated many times

  Expressed Recursively An optimal solution to a problem contains optimal solution to subproblems

- Subsequence: A sequence aIer deleting some elements
- A algorithm always makes the choice that looks best at the moment
  - Used for optimization
  - Make local optimal choices and hope to achieve global optimality (Greedy-choice property)
  - An optimal solution to the problem contains an optimal solution to subproblems (opCmal substructure)
- Graph
  - a symbolic representaBon of a network and of its connectivity
- There are two ways to represent graphs
  - Adjacency list
  - Adjacency matrix
- Cut edge: Edge whose deletion will increase the number of connected components (Disconnect the graph)

1 Complexities

1.1 Space Requirements

Adjacency Lists For directed and directed $\Theta(V+E)$

Adjacency Matrix For directed and undirected $\Theta(V^2)$

1.2 Search Complexities

Breadth-First $\mathcal{O}(V+E)$

Depth-First $\mathcal{O}(V+E)$

2 Coloring Depth First Search

White undiscovered

Gray discovered but not finished

Black finished (found every reachable vertex from it)

1 Notes

- A shortest may not exist when:
  - Negative weight cycle
  - The graph is not connected
- Two Properties of The Shortest Path
  - The optimal substructure of dynamic shortest path: A subpath of a shortest path itself is a shortest path
  - Triangle inequality
    * For all $u, v, x \in V$ we have $\delta(u, v) \leq \delta(u, x) + \delta(x, v)$
    * Shortest path from u to v is at most any particular path, e.g., the blue chain.
- Tractable Problems
  - Problems solvable in a polynomial time
- $P \subset NP$
- NPC: If any other problem (i.e., NP-Hard problems) p in NP is polynomial reducible to problem q

2 Psuedocode

2.1 Dijkstra(G,s)

```
d[s] = 0
for each vertex v in V—{s}
    d[v] = infinity
    S = emptySet
    Q = V

while Q is not empty
    u = Extract_Min(Q)
    S = S ∪ {u}
    for each vertex v adjacent to u
        if d[v] > d[u] + w(u,v)
            d[v] = d[u] + w(u,v)
```

2.2 Bellman-Ford

```
Given G=(V,E,w) and source vertex s
d[x] = distance estimate from s to x

Bellman-Ford(G,s)
d[s] = 0
for each vertex v in V-{s}
    d[v] = infinity

for i=1 to |V|-1
    for each edge (u,v) in E
        if d[v] > d[u] + w(u,v)
        d[v] = d[u] + w(u, v)

for each edge (u, v) in E
    if d[v] > d[u] + w(u, v)
        report that a negative-weight cycle exists
```

2.3 Floyd-Warshall

```
D^(0) = W
for k = 1 to n do
    for i = 1 to n do
        for j = 1 to n do
```

$d_{ij}(k) = \min\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\};$

```
Return D^(n)
```

2.4 Ford Fulkerson

```
FORD-FULKERSON(G, s, t)
    for each edge (u,v) in E(G)
        do f[u, v] = 0
           f[v, u] = 0
    while there is a path p from s to t in the residual network Gf
        cf(p) = min{cf(u, v)-f[u, v]: (u, v) is in p}
            for each edge (u, v) on p
                if (u, v) in E
                    f[u, v] = f[u, v] + cf(p)
                else f[v, u] = f[v, u] - cf(p)
```

3 Complexities

Dijkstra's $\Theta(V) \times T_{\text{Extract-Min}} + \Theta(E) \times T_{\text{Decrease-Key}}$

Bellman-Ford $\mathcal{O}(VE)$

Floyd Warshall $\mathcal{O}(V^3)$

Ford-Fulkerson $\mathcal{O}(E|f^*|)$

Edmond Carp $\mathcal{O}(VE^2) = \mathcal{O}(V^5)$

CS3100

Software Engineering I

Camelot Protocol Document

A description of how the system protocol will behave. Created and maintained by Ian Howell, Hunter Mathews, Illya Starikov, William Thurman, Zachary Wileman. *Server Team #1*

Last Revised: *March 14, 2017*
ILLYA STARIKOV

Michael Gosnell
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Status: **Validated**
Version: **1.1**

Contents

In essence, our server will act as an IRC-esque chatroom. Because we are starting with the Minimal Viable Product (MVP), it will not particularly have all the features of an IRC server; also, because we have to provide a JSON-based API, that alters the design drastically. This document will provide a sufficient outline on the authentication process, the basic objects (user, channel, message), and other miscellanies. Required fields are marked by a red asterisk (*), and possibly required items are marked with a blue asterisk (*).

The server will be asynchronous. The messages can be sent from the client at anytime, and will be handled accordingly. The messages should also be able to be received at anytime. It is up to the client to add listeners to be able to receive said messages.

1 Authentication

At the time of the authentication (i.e. the login process) a few item will have to be provided.

1.1 Input

Server Location* The server location must be provided to get access to the server. Duh. **String.**

Server Password* An optional password might be required. **String.**

User* A unique identifier to represent the user. If submitted, and not unique to channel, a randomized one will be returned. Can only consist of a mixture of numbers $[1 \cdots 9]$, letters $([a \cdots z]|[A \cdots Z])$ or underscores (_). **String. String.**

Channels *After* authentication (i.e. after successfully joining the server), channels can be joined. Refer to Section 2 for details. **Array of Objects `[{ Channel, Password }, ⋯ ]`**

1.2 Output

Session Key Will be used in future development. **Array.**

Username Will be used to verify if submitted username was allowed. **String.**

Channel List Success Will return an a array of objects to specify the success of joining the channels. **Array of Objects `[{Name, Success Status, [Users, ⋯] }, ⋯]`.**

2 Channel

The channels is the actual location of the chat room. All messages posted by the user(s) will appear here. A channel can act as a direct message mechanism as well, by simply having two users.

Name The name of the channel. The name of the channel must be unique. Can only consist of a mixture of numbers $[1 \cdots 9]$, letters $([a \cdots z]|[A \cdots Z])$ or underscores (_) **String.**

Password If there a password, specify it. If there is no password, any submissions (including SQL injections) will be accepted. **String.**

3 Message

A message is a simple object with what you'd expect a message to have. Simple message object will be sent and received (with all the contents below).

Timestamp The time *the client* sent this message. A JavaScript format time object should be sent. **String.**

Sender* The *username* of the sender. **String.**

Message Actual message contents. No format characters will be accepted (i.e. `\t`, `\n`, or `\r`). **String.**

Channel What channel the message was posted on. **String.**

4 Edit(s)

This section is dedicated compatibilty-breaking edit(s).

Illya Starikov – Anonymity Discontinued

Not only is true anonymity hard to implement and frustrating to maintain, a lot of the time it is only a hassle for users and programmers. The only people gaining from anonimity are typically trolls or abusers of the system. From this point forward, the bare minimum one can provide is a username. It is to the clients if they wish to make a randomized ID, but it should be displayed. *Changed on February 28, 2017.*

Camelot
System Requirements

All relevant documentation for system requirements. Created and maintained by Ian Howell, Hunter Mathews, Illya Starikov, William Thurman, Zachary Wileman. *Server Team #1*

Last Revised: *March 25, 2017*
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Status: **Validated**
Version: **1.2**

Contents

1 Introduction

Camelot is JSON-based server written exclusively in Python 3. The purpose of Camelot is to provide an IRC-esque chatroom on the server, leaving implementation details for the client up open to interpretation.

1.1 Purpose

This document is intended to guide development of Camelot. Not only will it provide the requirements (i.e. system requirements, user requirements, and interface requirements), but it will provide a decent outline for what the server entails (i.e. functions, constraints, dependencies, etc.). Understanding the system requirements will give a more clear and concise understanding of the Application Programming Interfaces (APIs) and their purpose.

1.2 How to Use This Document

Because this document will be reviewed by various different skill sets, this section will break down which parts should be reviewed by which type of reader.

1.2.1 Types of Reader

This document is going to be aimed at two different type of people: developers and end-users. The first set, client/server side developers, will be the Python3 Programmers, server-side programmers, Information Technology (IT) personal, and others that might have communication needs. These people are going to be the ones who wish to build upon or plan to use the code commercially. The other set will be the end-users. These people will be running the server in order to hold a chat session(s).

1.2.2 Technical Background Required

Programming competency *is imperative* to understanding the documentation. Programming methodology, jargon, and general concepts will be assumed in subsequent sections. Server side programming is recommended, but not required.

The document will stay to a high level server design. For the purpose of this document, source code will not appear. Specific tools/programming languages capability is not required.

1.2.3 Overview Sections

To get a a general understanding of the document, this following sections and subsections should be (chronologically) read:

1. User Characteristics (Section 2.3)
2. Assumptions and Dependencies (Section 2.5)
3. Specific Requirements (Section 3)

1.2.4 Reader-Specific Sections

Types of readers are described as follows.

- Server-side Developers: This would entail anyone interested in working on this particularly instance of the server. All of this document should be read.
- Client-side Developers: This would entail all who would want to get involved in creating a client. The following sections should be read:
  1. Description (Section 2)
  2. Specific Requirements (Section 3)
- End Users: This would entail anyone that is generally interested, but has no intentions of truly using it standalone.
  1. Product Perspective (Section 2.1)
  2. Product Functions (Section 2.2)
  3. User Characteristics (Section 2.3)

1.3 Scope of the Product

Truthfully, the intention of this project is to get an A in software engineerings. But I actually have to put something here, so.

There is an expectation at the end of the development lifecycle to open source this server. After open sourcing, the server team hopes that Camelot will serve as a model for server-side programming. Because the JSON-based framework would be familiar to people, the team hopes for a some market adoption. Overall, this would be a good model for how a simple server would be maintained.

1.4 Business Case for the Product

There is a real pandemic: there are not enough chat applications. Sure, there's Messenger, Slack, Skype, Viber, WeChat, WhatsApp, Line, GroupMe, Snapchat, Voxer, but they are garbage. The market needs a great server to tackle on these giants, and Camelot will do that.

1.5 Overview of the Requirements Document

1. Raspberry Pi 3 needed to host the server
2. Knowledge of Python to program the server
3. JSON based framework

2 Description

This section will give the reader an overview of the project, including why it was conceived, what it will do when complete, and the types of people we expect will use it. We also list constraints that were faced during development and assumptions we made about how we would proceed.

The Camelot Server project will create a means of communication between chat clients, in order to allow end users to communicate with each other.

2.1 Product Perspective

We have chosen to develop this project in order to create a standard form of communication between the chat clients that are being developed to use the server. The developers will use this in order to create an interface to house the information being transmitted.

2.2 Product Functions

The server will have the following capabilities.

2.2.1 User Authentication and Login

Users Will have the option to either create an account or to log in with an existing account.

Users will be able to create accounts by submitting a username that hasn't been taken. The client will send a JSON encoded file that looks something like:

```
{
    "create_account": {
        "username": "some username",
        "password": "some password",
        "server_password": "some server password"
    }
}
```

If the username has already been taken, then the client will receive a JSON encoded file that looks something like:

```
{
    "error": "That username is already taken."
}
```

When the client sends a request to login, it should send a JSON encoded package to the server. It should look something like this:

```
{
    "login": {
        "username": "some username",
        "password": "some password",
        "server_password": "some server password"
    }
}
```

If the username doesn't exist or if the password the user enters doesn't match the password associated with the user, then the client will receive a JSON encoded file that looks something like:

```
{
    "error": "The username and/or password do not exist in the database."
}
```

There will also be a server password given to each client (as seen above) that the user doesn't have to enter but the client will have to pass along with the user login so that the client can be authenticated.

2.2.2 Channels

Channel Creation

As of right now, there will be a set number of channels created by the server that the clients will have the option of joining. Later on in development, we may add the option for users to create channels. Upon creation, the creator will become the admin of that channel. Each user will have a limit on the number of channels they may create.

Channel Deletion

As of this writing, only the server team will have the option of deleting specific channels. When users gain the ability to create channels, they will also gain the ability to delete channels that they have created[1].

Initial Joining of Channels

After a user has logged in, the server will send the client a list of default channels that the user has the option of joining[2]. It will look something like this:

```
{
    "channels": [
        "Client␣Team",
        "Server␣Team",
        "Software␣Eng."
```

[1]The initial channels will be owned by the server team.

[2]Later on, the user will have the option to search for channels based on a keyword.

```
    ]
}
```

The client will then need to send back a JSON encoded file to the server describing what channels the user would like to join. The JSON file should look something like this:

```
{
    "join_channel": [
        "Client Team",
        "Server Team"
    ]
}
```

After the user successfully joins the channels, the specified user will have access to the channels that they decided to join.

Joining Channels After Selecting Initial Channels to Join

Users will have the option to list the channels when logged in. They will be able to join channels in the same fashion as they did initially[3].

2.2.3 Send/Receive Messages

Whenever the server receives notice that a new message has been posted to a given channel, the server will send out a message to each user who is connected to that given channel. The message will be a JSON file that looks something like this:

```
{
    "new_message": {
        "channel_receiving_message": "Client Team",
        "user": "username",
        "timestamp": "2017-03-14 14:11:30",
        "message": "the actual message that the user posted"
    }
}
```

When a user (client) wishes to send a message to a certain channel, the JSON object should look something like this:

```
{
    "new_message": {
        "channel_receiving_message": "Client Team",
        "user": "username",
        "timestamp": "2017-03-14 14:11:30",
        "message": "the actual message that the user posted"
    }
}
```

Notice that both the JSON file being received by the client and the JSON being sent out by the server look exactly the same. This is done so that the server can simply

[3]Later on, the user will have the option to leave channels.

broadcast messages to all users without having to decode a JSON file server side. This will give the least amount of delay in messages being sent out to each user. Also, the user that sends the message out will also receive the message. It's up to the client in how they want to approach this situation.

2.3 User Characteristics

Most users will be developers who will make clients to interface with the server. They will be mostly of a more technical background, with education background involving computer programming. They will be interfacing with the server in order to create a client for end users to communicate with each other. They may encounter obstacles with reading messages if they have no experience with parsing JSON.

2.4 General Constraints

For the constraints of our server, we didn't have any specific constraints as to what Integrated Development Environment (IDE) each person used or the platform that they developed on. The only specific constraints that we have so far is that the server is going to be developed with Python3 using JSON for data transfer and also that each person use Git for collaborating on the development of the server. We have not run into any issues with making our server compatible with other software as of yet.

2.5 Assumptions and Dependencies

The assumptions made with the development of this project is that each person that will be working on the server has a working knowledge of some programming language (most likely C++) and is willing to learn python. Its also assumed that at least some people working on the server have some background knowledge as to how to program in python as to help others within the server group who aren't as familiar with Python.

3 Specific Requirements

The follow sections will go in-depth about the specific requirements of the Camelot system.

3.1 User Requirements

Our user base will be split across two use cases:

1. The end users, who will be referred to as the user, and

2. The client teams, who will be referred as the developers

3.1.1 End User Requirements

The user will need to have a client that can connect with the Camelot server. They will need to be able to communicate with other users across channels. They should be able to send and receive messages. Messages should be sent and received in a logical order.

3.1.2 Developer Requirements

The developers will require the ability to request a list of channels. The list will contain information regarding the channel names and the users currently in each channel. With a successful login, they should be able to create to create a unique user with a specific identifier. Users should be deleted upon signing out or error. The developers should be able to request message information, such as message text, senders, requested channel, and timestamps.

3.2 System Requirements

The Camelot server will be run on a Raspberry Pi 3 Model B. It will need need to have internet access. The hardware will need to have Python3 installed as well as PostgreSQL for database management.

3.3 Interface Requirements

The Camelot will need to interface with a client using JSON formatting as a data transfer protocol.

4 References

Below is a list of all relevant documentation/references for this document.

C++ http://www.cplusplus.com

Git https://git-scm.com

JSON http://www.json.org

PostgreSQL https://www.postgresql.org

Python3 https://www.python.org/download/releases/3.0/

5 Version History

Below are major releases and their respective changes.

- **1.0** `Initial release`
- **1.1** `Bug Fixes`
  - `Fixed typos, mistakes, inproper wording, etc`
  - `Fixed Pandoc spacing issue where a \n would appear after 80 characters`
  - `Further explained functionality`
- **1.2** `Added verification chapter`

6 Server/Client Validation

The following signatures from partnering client teams will signify the approval of requirements document. This means approval of, but not limited to,

- The API guidelines
- The conceptual design
- The goals and vision for the product

The one signature below is representative of everyone in the group. Signee is held accountable for entirety of client team. Terms and condition apply. Limit to one daily. Side effects include upset stomach, indigestion, and spontaneous combustion.

Server Team #1	Client Team #1	Client Team #7
....	....	....
Signature	*Signature*	*Signature*
....	....	....
Date	*Date*	*Date*

7 Glossary

APIs Application Programming Interfaces. 3

C++ A general-purpose, object oriented programming language. 9, 11

Git A free and open source distributed version control system designed to handle everything from small to very large projects with speed and efficiency. 9, 11

IDE Integrated Development Environment. 9

IT Information Technology. 3

JSON JSON is a syntax for storing and exchanging data, written with JavaScript object notation. 4, 6–11

PostgreSQL Sophisticated open-source Object-Relational DBMS supporting almost all SQL constructs. 10, 11

Python An interepreted programming language popular for server side programming. 3, 9–11

Camelot Final Summary

A summary of the Camelot server. Created and maintained by Ian Howell, Hunter Mathews, Illya Starikov, William Thurman, Zachary Wileman. *Server Team #1*

Last Revised: *May 5, 2017*
Illya Starikov

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Status: **Approved**
Version: **1.0**

For our project, we decided to implement an asynchronous server written in Python3 and uses a PostgreSQL database to store user and channel information. The Python module `pytest` was also used for writing unit tests so that we may try to have have full code coverage on the project and "bring to light" any bugs that may exist in our project in terms of the Server and Database class files.

Some of the main accomplishments that were made on this project include: designing a database which was used for storing the user and channel information, creating a class file for both the server and database where the server file was dedicated to server functions (most functions mainly used by clients) and the database file was dedicated to interacting with the PostgreSQL database, writing unit tests to help better test our code so we could find any bugs/problems that may have existed, and created a multi-threaded asynchronous server for use with clients along with a client file used for testing purposes.

When designing the architecture, we decided to follow a Model View Controller paradigm. The controller was the server, the model was the database, and the view was the respective Client teams graphical user interface. In true MVC fashion,

- the server acts as the bridge between the client and the database
- the database stores all the data
- the client provides a view for the data

Our server runs on an asynchronous model. This means that we are able to service multiple clients independently of each other. This is accomplished by making use of Python3's threading library. Each client is thrown onto its own thread, and each thread is thrown into a list of threads. When a client sends a message, we loop over each client (within the respective channel), sending the message to each. This is extended to all other actions performed by a client, such as logging in and deleting channels. When a client closes the connection to the server, it is removed from the client list.

The database we designed uses three tables, one for the user, one for the channel, and one for the user in channel relationship. The `USER` table stores the username and password of the user. The `CHANNEL` table stores the channel name and the admin of the channel. The `CHANNELS_JOINED` table stores which users are in which channels. Not only does the database allows us to query which users should receive a message, but it also allows us to perform other more complex operations such as notifying all users when a channel is deleted. An additional PostgreSQL file was also included that adds some initial channels to the database that all users are, by default, added to when they create their account.

While we waited for the client teams to finish their clients and begin testing the connection to our server, we created a handful of unit tests that would allow us to test certain functions that we would provide to the client teams to use. These functions included creating accounts, logging in, creating channels, deleting channels, deleting accounts, sending messages, along with many other things. For each different test, we covered all of our bases, both fail and success. Some examples include:

- If the user was not logged in, then what would the server tell the client
- If there was an invalid JSON file, then what would the server tell the client
- If the function was to fail, make sure the server handled the exception properly

As far as any deviations from the requirements document, none seemed to occur for our team, but one of our client teams did experience a change to their requirements document that affected us. Specifically, one of the client teams was originally wanting us to implement private messaging as a feature but the team decided to abandon that idea after they found out they werent going to be able to implement this feature. So, due to that decision, the team decided to no longer pursue the implementation of the private messaging feature.

CS3200

Numerical Methods

1 Introduction

- The differential equation for newton's equation is $\frac{dv}{dt} = g - \frac{cv}{m}$, where cv is the drag coefficient $\times$ velocity.

$$\frac{dv}{dt} = g - \frac{cv}{m} \tag{1}$$

$$dv = (g - \frac{cv}{m})dt \tag{2}$$

$$\int \frac{1}{g - \frac{cv}{m}} dv = \int dt \tag{3}$$

$$\lg(g - \frac{cv}{m}) = t + c \tag{4}$$

$$v = mg/c(1 - e^{c/mt}) \tag{5}$$

2 Error Tolerance

- There will be 15% extra credit.
- Two important terms, accuracy & precision.
  - Accuracy refers to how close the computed value is to the true value.
  - Precision means how close the values are together.
- Because we want everything relative, we calculate with two different errors.
  - Absolute error $= e_a = |\text{True value} - \text{Approximation}|$
  - Relative error $= \epsilon_t = \frac{\text{True value} - \text{Approximation}}{\text{true}} \times 100$
- These equations can be used for precision as well.
  - $e_a = |\text{previous} - \text{current}|$
  - $\epsilon_t = \frac{\text{previous} - \text{current}}{\text{current}} \times 100$
- Stopping point is when it's below a certain threshold.
- If true value is 0, shift over by 1.
- For a result to be accurate or precise, should be $< 0.5 \times 10^{2-n}$, where n is the number of significant digits.
- the function of e^x can be represented as $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$
  - for $x = 1$, $e^1 = 2.71828$
  - for $x = -1$, $a = 1 - 1 = 0$

4 Roots of Equations

- The bisection method is one way to find the root.
- There are two requirements.
  1. $f(x)$ is continuous on $[a, b]$
  2. $f(a) \times f(b) = -(\text{something})$

Numerical Methods Crib Sheet
Illya Starikov

Integration Equations

Note that h usually refers to $(b-a)$. Also, Trapezoidal needs 2 points, Simpson's 1/3 uses 3, Simpson's 3/8 uses 4 and Boole's 5. Also note that for something like $[0,4]$,
$h=4, n=1, h=2, n=2, h=1, n=4$

`Trapezoidal`	$I = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n} + f(x_n)\right]$
`Richardson`	$I \approx {}^4/_3 I(h_2) - {}^1/_3 I(h_1)$
	$\approx {}^4/_3 I(\text{current cell}) - {}^1/_3 I(\text{previous cell})$
`Romberg`	$I_{j,k} = \frac{4^{k-1} I_{j+1,k-1} - I_{j,k-1}}{4^{k-1}-1}$
	$I_{j,k} = \frac{4^{\text{cell your on - 1}} I_{\text{cell your on}} - I_{\text{cell one up}}}{4^{k-1}-1}$
`Simpson 1/3`	${}^h/_6(f(a) + 4f(a+h) + f(b))$
`Simpson 3/8`	$3\,{}^h/_8(f(a+h) + 3f(a+h) + 3f(a+2h) + f(b))$

Differentiation Equations

Forward Finite-Divided

$$\frac{d}{dx} = \frac{f(x+h) - f(x)}{h} \quad O(h)$$

Backward Finite-Divided

$$\frac{d}{dx} = \frac{f(x) - f(x-h)}{h} \quad O(h)$$

Central Difference

$$\frac{d}{dx} = \frac{f(x+h) - f(x-h)}{2h} \quad O(h^2)$$
$$\frac{d^2}{dx^2} = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \quad O(h^4)$$

Richardson Extrapolation

$$I = I(h) + \mathcal{E}(h)$$
$$I = I(h_1) + \mathcal{E}(h_1) = I(h_2) + \mathcal{E}(h_2)$$
$$\mathcal{E} \approx -\frac{b-a}{2} h^2 \bar{f}''$$
$$\frac{\mathcal{E}(h_1)}{\mathcal{E}(h_2)} \approx -\frac{\frac{b-a}{2} h_1^2 \bar{f}''}{\frac{b-a}{2} h_2^2 \bar{f}''} \approx \frac{h_1^2}{h_2^2}$$
$$\mathcal{E}(h_1) \approx \mathcal{E}(h_2)\left(\frac{h_1}{h_2}\right)^2$$
$$I \approx I(h_1) + \mathcal{E}(h_2)\left(\frac{h_1}{h_2}\right)^2 \approx I(h_2) + \mathcal{E}(h_2)$$
$$\mathcal{E}(h_2) \approx \frac{I(h_2) - I(h_1)}{({}^{h_1}/_{h_2})^2 - 1}$$
$$I \approx I(h_2) + \approx \frac{I(h_2) - I(h_1)}{({}^{h_1}/_{h_2})^2 - 1}$$
$$I \approx {}^4/_3\, I(h_2) - {}^1/_3\, I(h_1)$$

Differential Equations

Midpoint Method

Note that $y(a) = b \implies x_i = a, y_i = b$.

$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_1 + {}^h/_2, y_i + k_1 {}^h/_2)$$
$$y_{i+1} = y_i + k_2 \cdot h$$

Heun Method

$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + h, y_i + k_1 h)$$
$$y_{i+1} = y_i + \frac{h \cdot (k_1 + k_2)}{2}$$

RK-3

$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + {}^h/_2, y_i + k_1 {}^h/_2)$$
$$k_3 = f(x_i + h, y_i + (-k_1 + 2k_2)h)$$
$$y_{i+1} = y_i + h \cdot (k_i + 4k_2 + k_3)/6$$

RK-4

$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_1 + {}^h/_2, y_i + k_1 {}^h/_2)$$
$$k_3 = f(x_i + {}^h/_2, y_i + k_2 {}^h/_2)$$
$$k_4 = f(x_i + h, y_i + k_3 h)$$
$$y_{i+i} = y_i + h^{(k_1 + 2k_2 + 2k_3 + k_3)/6}$$

Numerical Methods Equations

Illya Starikov

July 27, 2025

1 Taylor Series

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots \tag{1}$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{2}$$

2 Taylor Series of e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots \tag{3}$$

$$\frac{1}{e^{-x}} = \frac{1}{1 - x + \frac{x^2}{2} + \cdots} \tag{4}$$

3 False Position

$$r = b - \frac{(b-a) \cdot f(b)}{f(b) - f(a)} \tag{5}$$

4 Newton's Method

$$x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)} \tag{6}$$

5 Secant Lines

$$x_3 = x_1 - \frac{(x_2 - x_1) \cdot f(x_1)}{f(x_2) - f(x_1)} \tag{7}$$

6 Modified Secant Method

$$x_2 = x_1 - \frac{f(x_1)\Delta}{f(x_1 + \Delta) - f(x_1)} \tag{8}$$

CS3800

Operating Systems

1 Introduction

- Interrupts can come about by many sources
  - Keyboard, system clock, etc.
- Interrupts do not effect exceptions
- There is a bit in the PSW to determine if user/superuser mode

2 An Overview

- Horizontal — Generalized
- Vertical — Specialized

- A forked process may share a Run-Time Stack with its parent. False.
- A forked process may share code space with its parent. True.
- A forked process shares its parent's data space at all times. False.
- A thread may share a File Descriptor Table with its parent. True.
- A thread may share a program counter with its parent. False.
- A thread may share code space with its parent. True.
- A thread may share data space with its parent. True.
- An acceptable solution for implementing mutual exclusion is to let the users disable interrupts before entering a critical section and enable them after leaving the critical section. False.
- Assume that you have globally defined. Show the correct way to do the the following.

```
struct Shared {
char Name[10];
int Value; } S1, S2;

thread_t tid;
void * RtnValue;
void * FooBar(void * arg);
```

1. Store "THREAD 1" in S1's Name field: `strcpy( S1.Name, "THREAD 1" );`
2. Pass S1 in the following call: `pthread_create(&tid,NULL,FooBar, void *S1);`
3. From within the function FooBar, print the Name that was passed inside the struct: `printf("The Name Received = %s\n", ((struct Shared*)arg)->Name);`
4. store 2 times the thread id into the Value field of arg: `((struct Shared*)arg)->Value = 2 * pthread_self();`
5. return the struct via a thr_exit: `pthread_exit( arg );`
6. Have main retrieve the returned value: Use the variable RtnValue since you don't know which thread has exit-ed. Even though `pthread_join()` can be used to wait for a specific thread; there is no way to wait for "any" thread by using the Pthreads library in Linux.
7. Have main print which thread exit-ed and the integer returned: Same.

- Can you build a system where mutual exclusion condition is eliminated? No. It is a resource constraint and most resources (if not all) require it.
- Can you suggest a strategy that can achieve the desired result in the previous question, i.e. no circular wait can occur in the system Assign a number to each resource and require that, at any moment, a process can only request a resource whose number is higher than that of any resource it is currently holding.
- Context switch means a process switching from a "blocked state" to "ready state". False
- Create a variable named bar which stores the output of the 'users' command. `bar=$(users)`
- Determine the output for these commands: variable="Dennis" echo "**$**variable"? `Dennis`
- Determine the output for these commands: variable="Dennis" echo '**$**variable'? `$variable`
- Different threads within a process may specify different dispositions for a specific interrupt True
- Even if mutual exclusion is not enforced on a critical section, results of multiple execution is deterministic (i.e. it produces the same results each time it runs). False.
- Explain under what circumstances a spinlock might be less efficient than the alternative. In uniprocessor systems, processes can use CPU one at a time. If a process is "busy waiting" for a long time, then a spinlock might be less efficient.
- Explain under what circumstances a spinlock might be more efficient than the alternative. 1. If you have more processor power than you need (e.g. in multiprocessor systems). Spinlock do not require context switch when a process must wait on a lock, and therefore it is more efficient. 2. when the locks are expected to be held for a short time, a spinlock might be more efficient (saves the context switch time, therefore, preferred).
- Give an example event for each transition to occur. (You need to give at least five example events). {create process}: user starts a new process or a program issues a `fork()` or a `pthread_create()` call. {cpu available}: the process currently occupying CPU stops running and the process in front of the READY queue is scheduled to run {time expires}: currently running process uses up its alloted time slot for this turn (timer interrupt). {event wait}: the process issues an I/O read {event completed}: DMA controller interrupts CPU signalling the completion of an I/O read {exit system}: process terminates or segmentation fault.
- Give an example of something that a process might do to initiate a voluntary context switch. Relate this answer to the Process State graph. When a process initiates an I/O Read/Write or event `wait()`. In the Process State graph, the transition is from Running State to `BLOCKED(Waiting)` State.
- Give an example of something that would cause a process to experience an involuntary context switch. Relate this answer to the Process State graph. When a process is interrupted by an external source such as a timer. In the Process State graph, the transition is from Running to Ready State.
- Mach OS is a microkernal. True.
- Give two reasons why we should not give the users the power to disable/enable interrupts in a multiprogrammed computer system. (i) they may (unintentionally) forget to enable them which means all (hardware & software) interrupts will be ignored. (ii) they may intentionally abuse this power and let their processes dominate the system resources.
- Given

```
int X[10];
some appropriate function MyFun
pthread_t Tid;
```

1. Write the call to `pthread_create()`, which executes `MyFun`, which accepts the array `X` as an argument. `pthread_create(&Tid, NULL, MyFun, X)`
2. How can the code segment `"if( i < y ) cout << foobar(i,y);"` be made to behave as if it were atomic? Answer the question by recoding the segment.

```
pthread_mutex_t m=1;
pthread_mutex_lock( &m );
if( i < y ) cout << foobar( i, y );
pthread_mutex_unlock( &m );
```

- How does a thread acquire RAM that is NOT shared with other threads in the task? Define local variables and use them only in this thread scope.
- How does a user process get privileged operations performed? The hardware allows privileged instructions to be executed only in supervisor mode. If an attempt is made to execute a privileged instruction in user mode, the hardware does not execute the instruction, but rather treats the instruction as illegal and traps to the OS and automatically switches to supervisor mode. When a user program does a system call, the switch to the supervisor mode is done automatically by the OS and the priviledged instructions that are needed by the system routine can be executed. Since the OS code is trusted, there is no harm done by giving access to privilidged instructions this way.
- I've got a variable named `flibble` which contains the string `tom Jerry harry`. Write a command which stores just `tom` into a new variable. `new=$(echo $flibble | cut -f1 -d' ')`
- If a process uses up its allocated time slot, a timer interrupt occurs and the process is placed in a BLOCKED Queue. False. It is placed in the READY queue.
- If mutual exclusion is not enforced in accessing a critical section, a deadlock is guaranteed to occur. False.
- If the shared resources are numbered 1 through N and a process can only ask for resources that are numbered higher than that of any resource that it currently holds, then deadlock can never happen. True.
- If there is no mutual exclusion condition for any resource in the system, then there is no possibility for deadlock. True.
- If we constrain the resource requests in such a way that no cycle can occur in the process-resource graph, then deadlock can be prevented permanently. True.
- In Producer/Consumer problem access to shared buffer must be done in a critical section, but access to "in" and "out" pointers doesn't need to be done inside a critical section. False. "in" and "out" are shared variables. They need to be accessed inside a critical section.
- In Readers/Writers problem, it is possible to write code which is functionally correct but may lead to the starvation of writers. However, it is impossible to write the code such that readers may starve instead of writers. False. It is possible to write such code.
- In a uniprocessor environment, threads waiting for a lock always sleep rather than spin. Why is this true (and reasonable)? In a uniprocessor system, only one thread can run at a time. If a thread is spinning for a lock, it is doing nothing but wasting CPU cycles while waiting for the lock to become available. Instead, it is better that they sleep (in the blocked queue) while waiting so that others can use the CPU. When the lock becomes available, OS will wake them up.
- In dining philosophers problem, which senario may lead to a deadlock situation? If the simulation code is written in such a way that each philosopher first acquires the right fork and then attempts to acquire the left fork, a deadlock may occur. Because it is possible that philosophers may enter a circular wait situation in which each one holds the fork on the right and tries to get the fork on the left (which will never happen).
- In dining philosophers problem, why examining and acquiring a fork is done inside a critical section? Explain by giving an example what may go wrong if critical section is not used. Each fork is shared by two neighboring philosophers. If it is not handled inside a critical section, both philosophers may think that they acquired the same fork and start eating using the same fork. This situation leads to an incorrect simulation.
- In the specification of `pthread_join()` in Linux, there is no way to wait for "any" thread (i.e. the call should specify a particular `thread_id` to join) True.
- Q. In the "zombie" state, the process no longer exists but it leaves a record for its parent process to collect. True.
- LINUX is designed as a monolithic kernel. True. But, with a new twist for expanding OS functionality: LINUX supports dynamically installable modules. A module can be compiled and installed on a running version of the kernel. This is accomplished by providing system calls to install/remove modules.
- Many modern O.S.s use a microkernel design. What does that mean? Microkernel is a small privileged OS core that provides process scheduling, memory management, and communication services and relies on other processes to perform some of the functions traditionally associated with the operating system kernel. It removes all nonessential components from the kernel, and implements them as system and user-level programs. The result is a smaller kernel. Amoeba, Chorus, Mach, and Windows/NT use microkernel design approach.
- Multiprogramming (having more programs in RAM simultaneously) decreases total CPU efficiency (in comparison to Uniprogramming or Batch processing). FALSE. It increases CPU efficiency (see question above)
- Name at least two important differences between a POSIX thread created through a `pthread_create()` call and a child process created through a `fork()` call. 1) Threads share data space and file descriptors with the other threads and the parent process while forked process doesn't 2) a forked process has a seperate and unique PID and hence a process control block (PCB) while the threads created by a process uses their parent's PCB. 3) it takes less time to create a new thread, less time to switch between two threads within the same process, less time to terminate a thread 4) Thread Library provides more control over the execution of concurrent threads through system calls such as `pthread_yield()`, `pthread_suspend()`, `pthread_kill()`, and `pthread_continue()`
- One of the solutions proposed for handling the mutual exclusion problem relies on the knowledge of relative speeds of processes/processors. False. No solution should rely on such assumptions
- The word 'mutex' is short for mutual exclusion.
- UNIX kernel is designed as a microkernel False. UNIX is a monolithic kernel, meaning that the process, memory, device, and file managers are all implemented in a single software module.
- Unlike time sharing systems, the principal objective of batch processing systems is to minimize response time. False. Principal objective for time sharing is to minimize response time. Whereas, the principal objective for batch processing is to maximize processor use.
- What are the 3 possible dispositions that a process may specify with respect to a signal (interrupt)? 1) ignore signal; 2) run the default signal handler provided by the OS; 3) catch the signal and run the user's signal handler.
- What are the important differences between a unix `fork()` and `pthread_create()`? `fork()`: child process has the separate data space with parent process, but they have the same code space. `pthread_create()`: threads share data space, code space, and os resources, but they have the unique thread ID, register state, stack and priority, PC counter.
- What are the main goals today in the design of operating systems? Convenience for user, efficient utilization of the computer resources (CPU, memory, I/O devices), and expandibility.
- What are the necessary conditions for a deadlock to exist. a) mutual exclusion; b) hold and wait; c) no preemption d) circular wait.
- What does `#!` signify in a shell script? It's a special notation which tells the shell, "After this mark, read the name of the script interpreter i want to use."
- What does it mean to say "`rand()` call is not MT-safe"? MT-Safe : MultiThread-Safe This means that the behavior of the function `rand()` is not stable when two threads try to use it at the same time. To get around this problem, we use MT-safe interfaces (such as `rand_r()`). However, `rand_r()` doesn't exist on some multithreaded operating systems (e.g. LINUX), therefore, you need to execute such calls inside a Critical Section to make sure that only one thread can call the function at any one time.

```
pthead_mutex_t rand_mutex;
pthread_mutex_lock( &rand_mutex )
  ;
int a = rand() % 5;
pthread_mutex_unlock( &rand_mutex
  );
```

- What does it mean to say that "`pthread_mutex_lock(...)` is a blocking call"? If the mutex lock requested by this call is held by another process, the called will be blocked until the lock is released by the current owner (via executing a `pthread_mutex_unlock()` call)
- What does it mean to say that a library or a module is MT-Safe? MT-Safe: MultiThread-Safe which means that the behavior of the function is stable when two threads try to use it at the same time. In other words, a library/module protects its global and static data with locks and can provide a reasonable amount of concurrency.
- What does the command `du` do? figures out disk usage (size of a directory)
- What function is used from within a process to send a signal to a process? `kill()`
- What information is saved and restored during a context switch? Context switch requires saving the state of the old process and loading the saved state for the new process. Process state minimally includes current contents of registers, program counter, stack pointer, file descriptors, etc.
- What is a context switch? Switching the CPU to another process.
- What is a critical section i.e. what makes a section "critical"? In an asynchronous procedure of a computer program, a part that can not be executed simultaneously with an associated critical section of another asynchronous procedure. In general, a code segment in which shared variables, shared file descriptors (shared resources) are accessed is considered a critical section.
- What is a spinlock? When a process is in its critical section, any other process that tries to enter the same CS must loop continuously in the entry code of the CS until the first one gets out. The spinlock is the most common technique used for protecting a CS in Linux. It is easy to implement but has the disadvantage that locked-out threads continue to execute in a busy-waiting mode. Thus spinlocks are most effective in situations where the wait-time is expected to be very short.

```
spin_lock(&lock) /* Critical
  Section */ spin_unlock(&lock)
```

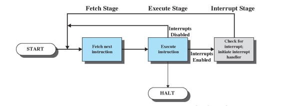
- What is an alternative to spinlocking? Put the process in a wait queue, so it doesn't waste CPU cycles and allow it to sleep until the block is released.
- What is an atomic operation? An operation that can not have it's execution suspended until it is fully completed. Generally, it is a single operation that can not be interrupted or divided into smaller operations.
- What is dual mode operation? User mode vs. supervisor mode.
- What is the difference between `pthread_mutex_lock` and `pthread_mutex_trylock`? `pthread_mutex_lock` is a blocking call. If we try to lock a mutex that is already locked by some other thread, `pthread_mutex_lock` blocks until the mutex is unlocked. But `pthread_mutex_trylock` is a nonblocking function that returns if the mutex is already locked.
- What is the difference between starvation and deadlock? STARVATION: A condition in which a process is indefinitely delayed because other processes are always given preference. DEADLOCK: An impasse that occurs when multiple processes are waiting for the availability of a resource that will not become available because it is being held by another process that is in a similar state.
- What is the effect of executing `pthread_yield()`? `pthread_yield()`: stop executing the caller thread and yield the CPU to another thread
- What is the required function prototype for the function `MyFun()`? `void* MyFun( void* X )`
- What was the original purpose/goal in the design of operating systems? To increase the efficiency of hardware.
- What would you expect to see (what instructions) surrounding the 'critical section' code?

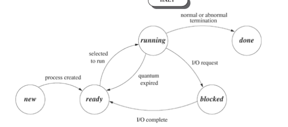
```
pthread_mutex_t mp;
pthread_mutex_lock( &mp );
// CRITICAL SECTION
pthread_mutex_unlock( &mp );
```

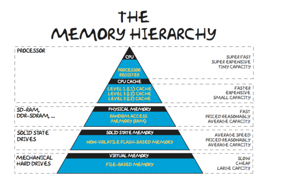
- When a process resumes execution after returning from a `fork()`, how can it tell if it is the original process or the new one? By the process ID which is returned by the `fork()` call. If it is equal to 0, it is the new one (child process); if it is a non-zero positive value, it is the original process.
- Which of the following strategies are used for deadlock prevention (circle all that apply)? A. Processes request all of the resources they need at once. They either get it all or get nothing and wait until they are all available. B. If a process needs to acquire a new resource, it must first release all resources it holds, then reacquire all it needs C. Remove the mutual exclusion condition on all of the resources D. Do not let any process hold more than 2 resources at any time. E. Resources are numbered sequentially in a total order. A process X can only ask for Rj if Rj $>$ Ri $\forall$Ri that X is currently holding. A, B, E.
- Which of the following would not necessarily cause a process to be interrupted? (a) Division by zero (b) reference outside user's memory space (c) page fault (d) accessing cache memory (e) end of time slice (f) none of the above d.
- While DMA (Direct Memory Access) is taking place, processor is free to do other things. The processor is only involved at the beginning and end of the DMA transfer. True.
- Who are the 2 individuals generally credited with the invention of C/Unix? Ken Thompson and Dennis Ritchie.
- Why are context switches considered undesirable (to be minimized) by OS designers? Because context switches waste a considerable amount of CPU time when they save and load the state of the processes.
- Why does a machine need dual mode operation? To ensure proper operation, we must protect the operating system and all the programs and their data from any malfunctioning program. The protection is accomplished by designating some of the machine instructions that may cause harm as "privileged" instructions. Dual mode operations can protect the OS from errant users, and the users from one another. It also prevents the abuse of priviledged instructions (such as 'interrupt enable/disable') by the user programs.
- Write a command which lists the processes that are owned by or contain the word foo in their name. `ps -ef | grep foo`
- Write a command which reads from the keyboard and stores it in the variable kbinput `read kbinput`
- Write a loop which reads the file 'foobar' into the variable 'lineIN' one line at a time `cat foobar | while read lineIN  (do ... done)`
- Write a shell statement which divide $\frac{3}{2}$ and stores the answer in $average `average=$(echo '3 / 2' | bc)`
- Write a shell statement which first adds the numbers 5 and 7 and stores the result in `$sum`. `sum=$(echo '5 + 7' | bc)`
- Write an if statement which compares `$variable` to the integer 5 to see if it is equal `if [ $variable -eq 5 ]`
- Write the command which throws output and errors away while running the script foo.sh. `foo.sh >& /dev/null`
- Write the name of the system call for obtaining the process id of of a process in UNIX? `getpid()`
- Write the name of the system call for obtaining the process id of a process' parent in UNIX? `getppid()`
- If I issue the command `someprogram.sh textfile` how do i access `textfile` as a variable? `$1`
- whats wrong with this command: `PATH='cellardoor'`? `$PATH` is a reserved system variable
- In a single processor system, there is no real multitasking. CPU time is shared among running processes. When the time slice for a running process expires, a new process has to be loaded for execution. Switching from one process or thread to another is called context switch. Process context switch involves saving and restoring process state information including program counter, CPU registers and process control block which is a relatively expensive (in terms of CPU time) operation. Similarly, thread context switch involves pushing all thread CPU registers and program counter to the thread private stack and saving the stack pointer. Thread context switch compared to process context switch is relatively cheap and fast as it only involves saving and restoring CPU registers.
- A semaphore, s, is a nonnegative integer variable that can only be changed or tested by these two atomic (indivisible/uninterruptable) functions:

```
P(s) : [while(s == 0) {wait}; s=s-1;]
V(s) : [s = s + 1;]
     P0() { ...
     P(mutex);
     balance += amount;
     V(mutex);
     ... }

     int pthread_create(pthread_t *
       thread, const pthread_attr_t *
       attr, void *(*start_routine) (
       void *), void *arg);
     attr = { scope, detachstate,
       stackaddr, stacksize,
       inheritsched, schedpolicy } ||
       NULL
```







Kernel a portion of the operating system that includes the most heavily used portions of software. Generally the kernel is maintained permanently in main memory. The kernel runs in a privileged mode and responds to calls from processes and interrupts from devices.

Critical Section in an asynchronous procedure of a computer program a part that cannot be executed simultaneously with an associated critical section of another asynchronous procedure

Preemption reclaiming a resource from a process before the process has finished using it

Concurrent pertaining to processes or threads that take place within a common interval of time during which they may have to alternately share common resources.

Main Memory memory that is internal to the computer system- is program addressable and can be loaded into registers for subsequent execution of processing

Time Sharing the concurrent use of a device by a number of users

Privileged Instruction an instruction that can be executed only in a specific mode usually by a supervisory program

Nonprivilaged State an execution context that does not allow sensitive hardware instructions to be executed such as the interrupt disable and I/O instructions

Mutual Exclusion a condition in which there is a set of processes only one of which is able to access a given resource or perform a given function at any time. See critical section.

Starvation a condition in which a process in indefinitely delayed because other processes are always given preference Symmetric Multiprocessing (SMP) a form of multiprocessing that allows the operating system to execute on any available processor or on several available processors simultaneously

Shell the portion of the operating system that interprets interactive user commands and job control language commands. It functions as an interface between the user and the operating system.

Interrupt Handler a routine generally part of the operating system. When an interrupt occurs control is transferred to the corresponding interrupt handler which takes some action in response to the condition that caused the interrupt.

Batch Processing pertaining to the technique of executing a set of computer programs such that each is completed before the next program of the set is started

Secondary Memory memory located outside the computer system itself including disk and tape

Download source on my Github.

- If a desired page frame is not currently resident in RAM, A Page Fault occurs.
- If a memory management system uses dynamic partitioning, External fragmentation may occur.
- Since paging system uses fixed size-sized pages, internal fragmentation may occur.
- Swapping out a piece of a process (i.e. pages of a process) just before that piece is needed is called Thrashing
- The least recently used (LRU) page replacement strategy is based on the principle of temporal locality as opposed to spatial locality.
- The top four levels in the memory hierarchy, starting with the fastest, are: Registers; cache memory; RAM; Disk
- The two lowest layers in the 7-layer ISO Open Systems Interconnect (OSI) model are Physical and Data Link layers and their primary function is to provide signaling technology and frame management
- Two transport protocols, Transmission Control Protocol (TCP) and User Datagram Protocol (UDP), are defined and handled at the Transport Layer
- DMA (Direct Memory Access) is a form of I/O in which a special module controls the exchange of data between main memory and an I/O device. During this I/O transfer, CPU is free to do other computation.
- In which one of the following OSI layers Transmission Control Protocol (TCP) and User Datagram Protocol (UDP) are defined and implemented? a. Application b. Physical c. Transport d. Data Link e. Session c
- When we compare clusters with SMP (Symmetric Multiprocessors), which of the following are true (circle all that apply)? a. Clusters are easier to manage and configure b. Clusters take up less space and draw less power c. Clusters are better for incremental and absolute scalability d. Clusters are superior in terms of availability e. Clusters have superior price/performance c, d, e
- Which of the following are among the direct goals of process scheduling algorithms (circle all that apply): a. improve response time b. minimize interrupts c. improve throughput d. minimize page faults e. improve turnaround time for jobs f. increase memory efficiency a, c, e
- Which of the following features are specific to Real-Time OS? (circle all that apply) a. Small size b. Fast context switch c. Less user control d. Nondeterministic delays e. Fail-safe operation a, b. e
- Which of the following malicious software need a host program to operate? (circle all that apply) a. Logic Bomb b. Worm c. Zombie (bots) d. Trojan Horse e. Virus a, d, e
- Which of the following scheduling policies may cause starvation for certain jobs? (circle all that apply) a. First Come First Serve (FCFS) b. Feedback c. Round Robin d. Shortest Process Next (SPN) e. Shortest Remaining Time Next (SRT) b, d, e
- Which of the following strategies is not used in a Disk Scheduling Algorithm? a) First in first out (FIFO) b) Last in first out (LIFO) c) Shortest service time first (SSTF) d) Longest service time first (LSTF) e) Back and forth over disk (SCAN) d
- Which one of the following is not among the 7-layers defined for ISO Open Systems Interconnect (OSI) model ? a) Application b) Routing c) Transport d) Data Link e) Physical b
- Which one of the following is not among the set of events that may take place between the time a page fault occurs and the time the faulting process resumes execution? a) OS blocks the process and puts it into a wait queue. b) One of the processes in the ready queue is selected to run. c) A DMA is initiated to load the page from disk into main memory d) A page replacement strategy is used to find a page frame to load the new page e) Page table is updated to reflect the change. f) none of the above f
- Which one of the following is not among the set of events that may take place between the time a page fault occurs and the time the faulting process resumes execution? a) OS blocks the process and puts it into a wait queue. b) One of the processes in the ready queue is selected to run. c) A DMA is initiated to load the page from disk into main memory d) The last page that the faulting process was executing is replaced with the newly loaded page. e) Page table is updated to reflect the change. f) none of the above d
- What are the three popular strategies for allocating free memory blocks to processes in dynamic memory partitioning? Explain briefly how each strategy works First-fit: chooses the first free block in the list that is large enough for the request; Best-fit: chooses the free block that is closest in size to the request; Next-fit: chooses the first free block that is large enough for the request and comes after the 'Last Allocated Block' in the list
- What interrupt is created when a desired page frame is not currently resident in RAM? Page fault trap
- How does the hardware know that a desired page frame is not currently resident in RAM? Valid bit
- What precisely does it mean if the dirty bit is set for a page frame? The page frame has been modified
- What is good vs. bad program locality? Good locality means that the process executes in clustered pages. Bad locality means that the process executes in scattered pages
- Explain when/how internal fragmentation may occur When fixed-sized pages are used, the last page of a program may be partially filled. This is called internal fragmentation
- Explain when/how external fragmentation may occur Segmentation system breaks up the memory space into variable-sized pieces. After a sequence of allocation and deallocations, free memory may get fragmented into small pieces. Even if the total size of free memory is large enough to satisfy large memory requests, a large request may not be met due to the lack of continuity between small fragments. This is called external fragmentation. Compaction is needed to put free blocks into one large memory block
- What is a global allocation scheme? Global replacement allows a process to select a replacement frame from the set of all frames, even if that frame is currently allocated to some other process; one process can take a frame from another
- What is a working set model? The working set model assumes that processes execute in localities. The working set is the set of pages in the current locality. Accordingly, each process should be allocated enough frames for its current working set
- Comparing global allocation vs. working set allocation, which would be more adversely affected by a program with bad locality? and WHY would that be true? Working set allocation would be more adversely affected by a program with bad locality. This is because the program with bad locality has poorly defined working sets and therefore, many page faults are likely to occur
- What is the "largest" program that could execute on a machine with a 24-bit virtual address? 2^{24} byte
- What is the "largest" program that could execute on a machine with a 24-bit physical address? Can't tell. Need to know the size of the virtual (logical) address
- The address contained in a TLB entry ¡PTE¿ is (physical—logical) physical
- List at least 3 flags that are contained in a PTE Valid bit, Reference bit, Dirty bit
- Define hit-ratio in a memory management context in a two-level memory (cache-RAM or RAM-Harddisk), the fraction of all memory accesses that are found in the master memory (i.e. the cache)
- How does the kernel know where on disk the desired information is for a non-resident frame? If valid bit=0, Page Table Entry should contain the Disk address
- Describe what demand paging means The technique of only loading virtual pages into memory as they are accessed is known as demand paging. If the demand pages are not in memory, a page fault trap happens, and the operation system swaps them in
- Describe what prepaging means Prepaging brings in more consecutive pages than needed. If a virtual page X causes a pagefault, then virtual page (X+1) is also brought in along with X. It is less overhead to bring in pages that reside contiguously on the disk
- Explain what the following C calls do both when the call is successful and when it is unsuccessful. 1. `socket( AF_INET, SOCK_STREAM, 0 )` 2. `bind(sd, (struct sockaddr*)&server_addr, sizeof(server_addr))` 3. `socket( AF_INET, SOCK_DGRAM, 0 )` 4. `accept( sd, (struct sockaddr*)&client_addr, &client_len )` 1. creates an internet stream (TCP) socket and returns the socket descriptor. If the call fails, it returns -1. 2. Binds the definition of a socket (socket descriptor) to a port number. If the call fails, it returns -1. 3. creates an internet datagram (UDP) socket and returns the socket descriptor. If the call fails, it returns -1. 4. Blocks execution until a client connection is received. When that happens, it returns a descriptor for the connection. If the call fails, it returns -1
- What does an Internet Protocal do? 1. Provides a naming scheme which uses a uniform format for host addresses 2. Provides a delivery mechanism by defining a standard packet format
- What are the possible goals that any scheduling policy might try to accomplish (list at least three)? To improve response time, Turnaround time (TAT), Throughput, Processor Efficieny
- Which decisions are made by Long-term, Medium-term, and Short-term scheduling? Be brief Long-term scheduling determines which programs are admitted to the system for processing and controls the degree of multiprogramming. Medium-term scheduling determines which programs will be resident. Part of the swapping function. Swapping-in decision is based on the need to manage the degree of multiprogramming Short-term scheduling determines which program will be executed on CPU next. Known as the dispatcher Executes most frequently
- Name 3 things that are essential to launch a bot attack 1) attack software 2) a large number of vulnerable machines 3) locating these machines (scanning or fingerprinting)
- Dennis Richie and Ken Thompson are generally credited with the invention of C/Unix.
- Bill Gates and Paul Allen started Microsoft in 1975.
- Steve Jobs and Steve Wozniac co-founded Apple. Steve Jobs then started NeXT, and was the CEO of Pixar.
- MS/DOS was 90
- What person Ed Roberts what company MITS built the 1st commercially available personal computer in 1975?
- Gordon Moore is one of the Intel founders.
- World's first personal computer, Altair 8800, was designed by Ed Roberts and was introduced in 1975
- The first mass market PC company is Apple.
- What corp may fairly take credit for inventions like the mouse, windows, pull-down menus etc.? Xerox/PARC
- What did Steve Jobs see while visiting PARC that inspired him to build a different kind of computer? GUI
- What did Jobs see that he completely ignored? object oriented programming and E-mail.
- What was the 1st computer that Jobs built based on this inspiration (that flopped)? Lisa.
- What was the 2nd one that didn't flop? Macintosh
- What product got Microsoft into the microcomputer software business? BASIC language interpreter
- What lucky event got Microsoft into the operating system market? Gary Kildall didn't eagerly pursue IBM when they requested a new OS. His wife and attorney would not sign a nondisclosure agreement. Bill Gates of Microsoft saw this as an opportunity and jumped in.
- What company purchased NeXT and their OS NExTStep? What year? Apple, in 1996
- What is a killer application? Software that's so useful that people will buy computers just to run it.
- What was the killer app for the Apple II? Visicalc
- What was the killer app for the IBM PC? Lotus 1-2-3
- What was the killer app for the Apple MacIntosh? Wysiwyg - What you see is what you get -¿ Desktop Publishing
- Why didn't IBM create their own OS for their 1st PC? wanted to manufacture and market it very fast; within one-year "....Once IBM decided to do a personal computer and to do it in a year - they couldn't really design anything, they just had to slap it together, so that's what they did ..."
- Who should have sold IBM their operating system for the 1st IBM PC? Gary Kildall of Digital Research
- What was the one part of the 1st IBM PC that was proprietary (that Compaq had to later reverse engineer)? ROM-BIOS
- Why did IBM decide to build the PC using open architecture? To save time, instead of building a computer from scratch, IBM initially decided to buy PC components off the shelf and assemble them – in IBM terms, this was called an open architecture. IBM made some changes to this initial decision. What was the almost immediate result of IBM having made that decision? IBM had to buy the OS and other software from other companies as well.
- What was IBM's motivation for designing/building PS-2/OS-2? IBM planned to steal the market from Gates with a brand new OS called OS/2.

CS4096

Software Systems Development

Ethical, Security, Legal, and Social Impact

CS 4096, Dr. Morales

Team Splatoonio*

July 27, 2025

War Paint is an augmented reality application that utilizes the player's current geographic location as a stylus to compete against other users. As part of gameplay this application collects user data and displays some for other users to see. While applications that give others your current location exist (i.e., SnapChat or Find My Friends) none of these display your location to anonymous users. In order to play War Paint the user must accept that their current location is being shared with other users; this creates an ethical conundrum. First, it would be illegal to allow children 13 years or younger to use the application without parent permission. There is a legal gray area in sharing a user's location from age 13 to 18, so to mitigate these ethical concerns, the application will likely be restricted to users 18 years or older.

The creators of War Paint also have an ethical responsibility to preserve the security of user data. War Paint will likely collect user data for the purpose of targeted advertisement; however, this data will never be sent to the War Paint server. In other words, we do not permanently store user data for longer than is necessary to play the game, and the data we do collect is all anonymous. It is important that this, as well as user passwords and profile information, remain secure and confidential. Passwords are currently stored on the server in plaintext due to time constraints, but before release, the server would be switched over to using a standard salted hashing function to protect user data. Clients would also be forced to send communication over HTTPS ensuring their connections were not visible in transit. In addition, most, if not all, of our server endpoints are also vulnerable to SQL injection, and would be fixed before releasing to mobile app stores. We would also like to have someone perform a penetration test of our system to help verify its security.

New users would be required to read and accept an End-User License Agreement stating that they are at least 18 years of age. It will clarify that the War Paint mobile app will collect user data and display the user's current location anonymously to other

*Adam Evans, Illya Starikov, Ian Howell, Deacon Seals, Michael Harrington, Luke Parton, Eric Michalak

players while the game is in session. It will state that the creators and owners of War Paint are not responsible for damages or risks incurred by sharing the user's location.

The EULA will also clarify that users are responsible for their own physical safety while playing. War Paint is a physically active game. The most successful way to play is to maintain the highest possible speed for the duration of a game session. This would encourage especially competitive users to take risks in order to succeed. To prevent people from using cars or other motorized vehicles, a strict speed limit of 20 miles per hour will be imposed on the players. If this speed limit is exceeded, the user will cease painting the map until their speed falls back to acceptable levels. Repeatedly violating the speed limit will result in a temporary ban. In addition, there will be a report system that allows players to indicate which players are participating in dangerous behaviors while playing.

After witnessing the social impact Pokémon GO had on cities and parks, it is hard to estimate the potential impact of War Paint. On a smaller scale it could increase the number of people running on college campuses. Encouraging college students to exercise with their peers would be a positive social impact. On a larger scale this game could cause congestion in areas where it is the most popular. Matchmaking could keep track of the number of users in any specific area to prevent too many people from playing in too small a location.

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Reflection Essay

Senior Design (Comp Sci 4096)

Illya Starikov

Due Date: Wednesday, December 13th

Throughout the course of Senior Design, I have learned a lot of useful information that will translate not only to the remainder of school, but to the actual development in the real world. Along the way of creating War Paint, I have learned several lessons that I hope to take with me upon graduating college.

For backstory, War Paint is a real-time, augmented reality game who's primary objective is to traverse as much of a terrain as possible. Specifically,

1. The player is boxed in a particular perimeter. Traversing outside said parameter is strictly forbidden (and will not count towards team score).

2. Players are split into even teams. As a player moves around the boxed perimeter, "paint" is placed down (colored with the team's primary color, red or blue).

3. Paining over another team's paint voids the other team's paint, and contributes only to who's team has the last layer of paint on the field.

4. At the end of a designated period of time, the team with the most paint on the board wins.

During the semester, I got to work with a team that had to deal with a whole stack. Our particular team had

- An Android Team
- An iOS Team
- A R&D Team (to focus on potential of hardware)
- A Server Team

Because my specialty was iOS Development, I took the role of working on the iOS application. Most of the software development process was simple: design the Model View Controllers (MVCs). The MVCs already had a hierarchy to them, so barely any architecture work had to be done.

The takeaway I gained from this class wasn't a particular tool or methodology; it was a particular mindset. Most of my group projects up to this point have placed me in a leadership role of some sorts. Because of my busy schedule this semester, I was not comfortable in that position. Because of this, I had to learn how to coordinate with others, be a team player, and put my faith into a different team lead.

My knowledge of the Software Development Process has not particularly changed up to this point. I have held several internships and have worked on massive projects before; most of my knowledge had been learned outside of the classroom.

The most important lesson I learned was how to not bite off more than I can chew. I had to usually take the minimal amount of work per week, for I had priorities in other classes and my job. This will greatly impact the way I handle work, where a work-life balance will be hard to maintain. I hope to keep this lesson in the back of my mind for future reference.

CS5200

Analysis Of Algorithms

Homework #8

Analysis of Algorithms

Illya Starikov

Due Date: November 27^{th}, 2017

Contents

Listings

Question #1

Theorem 1. *We take the definition of $\mathcal{O}$ as such.*

For $f(n) \in \mathcal{O}\left(_n um_1\right)$, there must exist constants c, where $c > 0$, and there must exist $n_0 \in \mathbb{N}$, where $n_0 \geq 1$, such that

$$|f(n)| \leq c|g(n)| \;\forall n \geq n_0$$

Problem #1.1

Problem Statement. ***Without using limits, but only the definition of $\mathcal{O}$*** *prove that $81n^3 + 1300n^2 + 300n \in \mathcal{O}\left(_n um_1\right)$, but that $n^5 - 15000n^4 - 10n^3 \notin \mathcal{O}\left(_n um_1\right)$. Show all work.*

Proof. In this instance, $f(n) = 81n^3 + 1300n^2 + 300n$ and $g(n) \in \mathcal{O}\left(_n um_1\right)$. To prove that $f(n) \in \mathcal{O}\left(_n um_1\right)$, first we must find constants c and n_0.

Take the following, $c = 1$, $n_0 = 15\,000$. Then, for all $n \geq 15000$,

$$81n^3 + 1300n^2 + 300n \ll n^5 - 15000n^4 - 10n^3$$

We see this is true, because

$$\begin{aligned} f(n) &= 81n^3 + 1300n^2 + 300n \\ &\leq 81n^3 + 1300n^3 + 300n^3 \\ &\leq 1681n^3 \\ g(n) &= n^5 - 15000n^4 - 10n^3 \\ &\geq n^3 - 15000^3 - 10n^3 \\ &\geq -14991n^3 \end{aligned}$$

From this we, we see that

$$|f(n)| \leq c|g(n)| \;\forall n \geq n_0$$

Therefore, $f(n) \in \mathcal{O}\left(_n um_1\right)$, and $81n^3 + 1300n^2 + 300n \in \mathcal{O}\left(_n um_1\right)$. □

Proof. Suppose not. That is, suppose that $\exists c, n_0$ such that

$$|f(n)| \leq c|g(n)| \;\forall n \geq n_0$$

Therefore, c, n_0 must satisfy the equation

$$
\begin{aligned}
81n^3 + 1300n^2 + 300n &\geq c \times (n^5 - 15000n^4 - 10n^3) \\
n^5 &\leq 15000n^4 - 10n^3 + c \times (81n^3 + 1300n^2 + 300n) \\
&\leq \frac{15000}{n} + \frac{10}{n^2} + c \times \left(\frac{81}{n^2} + \frac{1300}{n^3} + \frac{300}{n^4}\right) \\
&\approx 1 + 3.6 \times 10^{-7}c
\end{aligned}
$$

From this, the inequality must be satisfied:

$$n \leq \max\left(n_0, \delta + 1 + 3.6 \times 10^{-7}c\right)$$

As we see, there is no values of n_0 and c that will make this inequality hold for all n. This has led us to our contradiction. Therefore

$$n^5 - 15000n^4 - 10n^3 \notin \mathcal{O}\left({}_num_1\right)$$

□

Problem #1.2

Problem Statement. *Let $f(x) = 2\sin^3(x) - 4\sin^2(x)$ and $g(x) = 2\cos^4(x) + 5\cos(x)$. Determine whether $f(x) \in \mathcal{O}\left({}_n um_1\right)$ or $g(x) \in \mathcal{O}\left({}_n um_1\right)$. You must show all work and base it directly on the definition of $\mathcal{O}$.*

Proof. Because both $\sin x$ and $\cos x$ are sinusoidal, we cannot compare them directly. For the purposes of this problem, we will use a Taylor Series expansions (Equation 1).

$$\sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{1}$$

For $f(x)$ and $g(x)$, we get the following expansions.

$$\sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = -4x^2 + 2x^3 + \frac{4x^4}{3} - x^5 - \frac{8x^6}{45} + \frac{13x^7}{60} + \frac{4x^8}{315} + \mathcal{O}\left({}_n um_1\right)$$
$$\sum_{n=1}^{\infty} \frac{g^{(n)}(a)}{n!}(x-a)^n = 7 - \frac{13x^2}{2} + \frac{85x^4}{24} - \frac{1093x^6}{720} + \frac{3329x^8}{8064} - \frac{263173x^{10}}{3628800} + \frac{839681x^{12}}{95800320} + \mathcal{O}\left({}_n um_1\right)$$

From this, we can clearly see that starting at the third term, $g(x)$ grows much more rapidly than $f(x)$. Furthermore, we see that the exponents of $g(x)$ grow by increments of 2, while $f(x)$ only grows by an increments of 1.

Therefore, choosing $c = 1$ and $n_0 = 1$,

$$\begin{aligned} |f(x)| \leq c|g(x)| \ \forall n \geq n_0 &\implies f(x) \in \mathcal{O}\left({}_n um_1\right) \\ &\implies 2\sin^3(x) - 4\sin^2(x) \in \mathcal{O}\left({}_n um_1\right) \end{aligned}$$

□

Question #2

For the proceeding problems, the source code is as follows.

Question #3

Problem #3.1

Problem Statement. *Let function q be as below:*

```
def q(n):
    if n <= 0:
        return 1
    elif n < 2:
        return 7
    else:
        return q(n - 1) + q(n - 2)
```

Let function sq be as below:

```
def sq(n):
    if n < 0:
        return 0
    else:
        return sq(n - 1) + q(n)
```

Conjecture a very simple linear relationship between q and sq.

After careful inspection of the sequences, it became quite apparent that there was a relationship between the two sequences is a constant and two terms in the sequence. In other words,

$$s(q) = q(n+2) - 7 \tag{2}$$

These findings can be summarized by Table 1.

Table 1: The values of $q(n)$, $sq(n)$, and $q(n+2)-7$

$q(n)$	$sq(n)$	$q(n+2)-7$
1	1	1
7	8	8
8	16	16
15	31	31
23	54	54
38	92	92
61	153	153
99	252	252
160	412	412
259	671	671
419	1090	1090
678	1768	1768
1097	2865	2865
1775	4640	4640
2872	7512	7512

Problem #3.2

Problem Statement. *Prove, using induction, that the relationship that you conjectured in the previous part is correct. Be sure to set your proof up correctly and to list explicitly the steps of a proof by induction.*

Proof. We will prove the following with induction. To prove this, first we must take into consideration that:

$$sq(n) = \sum_{i=0}^{n+1} q(n) = q(n+2) - 7 \tag{3}$$

Define The Problem For this problem, we wish to prove that $p\,{}_{n}um_1 \overset{\propto}{\underset{n}{\sim}} um_1\, q$ by the relation modeled by Equation 2. We map this relationship $\forall n \in \mathbb{Z}^+$.

Check Base Case Two Other Values Refer to Table 1 for the first three values, along with 12 more values.

Prove for all $n > s$, that if $P(n-1)$ is true, then $P(n)$ is true Assume the following inductive hypothesis:

$$\sum_{i=0}^{n-1} q(n) = q(n+1) - 7$$

Then we are going to prove

$$\sum_{i=0}^{n} q(n) = q(n+2) - 7$$

We prove so by such:

$$\begin{aligned}\sum_{i=0}^{n} q(n) &= \sum_{i=0}^{n-1} q(n) + q(n) \\ &= (q(n+1) - 7) + q(n) \\ &= q(n+2) - 7\end{aligned}$$

Conclude The proof Because we have proved the base case and the inductive step, we use inducion to conclude $sq(n) = q(n+2) - 7$.

$\square$

Question #4

Problem Statement. *Let Vec_n be the set of all vectors of length n each component of which comes from range(n). For example, $(3, 0, 2, 2) \in Vec_4$. If $v \in Vec_n$, a quirk is defined as a pair (i, j) such that $0 \leq i < j \leq n-1$, but $v[i] > v[j]$.*

Problem #4.1

Problem Statement. *Create a sample space consisting of the elements of Vec_3. List all the elements of this space. Turn it into a probability space by using the uniform distribution. Finally, compute the average number of quirks in members of Vec_3.*

The sample space S as follows,

$$\begin{gathered}
\text{S} = (0,\ 0,\ 0), (0,\ 0,\ 1), (0,\ 0,\ 2), (0,\ 0,\ 3), \\
(0,\ 1,\ 0), (0,\ 1,\ 1), (0,\ 1,\ 2), (0,\ 1,\ 3), (0,\ 2,\ 0), \\
(0,\ 2,\ 1), (0,\ 2,\ 2), (0,\ 2,\ 3), (0,\ 3,\ 0), (0,\ 3,\ 1), \\
(0,\ 3,\ 2), (0,\ 3,\ 3), (1,\ 0,\ 0), (1,\ 0,\ 1), (1,\ 0,\ 2), \\
(1,\ 0,\ 3), (1,\ 1,\ 0), (1,\ 1,\ 1), (1,\ 1,\ 2), (1,\ 1,\ 3), \\
(1,\ 2,\ 0), (1,\ 2,\ 1), (1,\ 2,\ 2), (1,\ 2,\ 3), (1,\ 3,\ 0), \\
(1,\ 3,\ 1), (1,\ 3,\ 2), (1,\ 3,\ 3), (2,\ 0,\ 0), (2,\ 0,\ 1), \\
(2,\ 0,\ 2), (2,\ 0,\ 3), (2,\ 1,\ 0), (2,\ 1,\ 1), (2,\ 1,\ 2), \\
(2,\ 1,\ 3), (2,\ 2,\ 0), (2,\ 2,\ 1), (2,\ 2,\ 2), (2,\ 2,\ 3), \\
(2,\ 3,\ 0), (2,\ 3,\ 1), (2,\ 3,\ 2), (2,\ 3,\ 3), (3,\ 0,\ 0), \\
(3,\ 0,\ 1), (3,\ 0,\ 2), (3,\ 0,\ 3), (3,\ 1,\ 0), (3,\ 1,\ 1), \\
(3,\ 1,\ 2), (3,\ 1,\ 3), (3,\ 2,\ 0), (3,\ 2,\ 1), (3,\ 2,\ 2), \\
(3,\ 2,\ 3), (3,\ 3,\ 0), (3,\ 3,\ 1), (3,\ 3,\ 2), (3,\ 3,\ 3)
\end{gathered}$$

The average number of quirks for $Vec_3 = 1$.

Problem #4.2

Problem Statement. *Generalize the results of Part (a) to determine the average number of quirks in members of Vec_n. You do not need to list the elements of Vec_n.*

For Vec_n, we have the following:

$$\text{Number of } quirks \in Vec_n = 0.25(n-1)^2$$

Question #5

Problem Statement. *Let G be defined by the following equations: $G(0) = 5$, $G(1) = 15$, $G(2) = 40$, and for $n > 2$, $G(n) = G(n-1) + G(n-2) + G(n-3)$. Write a Python program that implements G directly from the definition. Submit this program as a* `.py` *in your ZIP file. Try to compute $G(500)$ with this program. If you can't compute $G(500)$ directly show how to use dynamic programming to write a more efficient program. Submit this program in your ZIP file.*

Let H be defined by the following equations: $H(0) = 6$, $H(1) = 7$, $H(2) = 8$, and for $n > 2$, $H(n) = H(n-1) - H(n-2) + H(n-3)$. Write a Python program that implements H directly from the definition. Submit this program as a `.py` *in your ZIP file. Try to compute $H(500)$ with this program. If you can't compute $H(500)$ directly show how to use dynamic programming to write a more efficient program. Once you compute $H(500)$ see if you can discover a more efficient program. Submit all of these programs in your ZIP file.*

For the sample input, the following is generated.

$$
\begin{aligned}
G(500) &= 213\,546\,417\,395\,738\,934\,772\,794\,111\,784\,493\,777\,375\,698\,990\,926\,394\,537\,758\,958\,705\,709 \\
&\quad 75\,006\,915\,873\,346\,065\,610\,819\,405\,691\,957\,713\,899\,246\,806\,099\,708\,361\,322\,154\,885\,470\,915 \\
H(500) &= 6
\end{aligned}
$$

For a $\mathcal{O}\left({}_n um_1\right)$ solution to $H(n)$, the following was produced:

$$
H(n) = \begin{cases} 6 & \iff n \mod 4 = 0 \\ 7 & \iff n \mod 2 \neq 0 \\ 8 & \iff (n-2) \mod 4 = 0 \end{cases}
$$

Question #6

Problem Statement. *Suppose you are dealing with a dynamic table that follows the following rules:*

1. *The table size doubles when the table is full and another element is added.*
2. *The table contracts to 2/3rds of its size when its load factor falls below 1/3.*

Using the potential function

$$\Phi(T) = |2 \times T.num - T.size|$$

show that the amortized cost of a TABLE-DELETE for this strategy is bounded above by a constant. For the definition of all terms, consult your textbook.

Proof. We know the amortized cost of the ith operation to be

$$\hat{c}_i = c_i + \Phi_i - \Phi_{i-1} \tag{4}$$

Assuming the ith operation does not trigger a contraction, using Equation 4,

$$\begin{aligned}
\hat{c}_i &= c_i + \Phi_i - \Phi_{i-1} \\
&= 1 + (2 \times T.num_i - T.size_i) - (2 \times T.num_{i-1} - T.size_{i-1}) \\
&= 1 + (2 \times T.num_i - T.size_i) - (2 \times T.num_i - T.size_i) + 2 \\
&= 3
\end{aligned}$$

However, if the ith operation does trigger a contraction,

$$\begin{aligned}
\hat{c}_i &= c_i + \Phi_i - \Phi_{i-1} \\
&= T.num_i + 1 + (2 \times T.num_i - T.size_i) - (2 \times T.num_{i-1} - T.size_{i-1}) \\
&= T.num_i + 1 + \left(\frac{2}{3}T.size_{i-1} - 2 \times T.num_i) - (T.size_{i-1} - 2 \times T.num_i - 2\right) \\
&= 2
\end{aligned}$$

We see in either situations, the amortized cost of a TABLE-DELETE operation is bounded by $2 \leq \hat{c}_i \leq 3$.

□

Question #7

Problem #7.1

Problem Statement. *Consider the following two sets of coin denominations: {1 cent, 8 cents, 20 cents}, {1 cent, 6 cents, 18 cents}. Describe a greedy algorithm that will express any sum given in pennies in terms of the denominations given in a set. Determine whether the greedy algorithm always produces the optimal solution for these two sets or not. Give a convincing reason for your conclusion.*

A greedy algorithm that always produces a solution could be described as follows:

1. Take $S = \varnothing$ to be the solution set, *coins* to be the input of coins, and T to be the target to reach.
2. Sort *coins* in ascending order.
3. While $\sum_{x \in S} x \neq T$
   a) Take the difference δ to be $T - \sum_{x \in S} x$.
   b) Pick the largest coin c such that $c \leq \delta$[1].
   c) Add this coin to S.

Although this always produces a solution, it does not always produce an optimal solution. For example, suppose our target $T = 24$. With the algorithm described above, then it would take 5 coins ($1 \times 20¢ + 4 \times 1¢$), while the optimal solution would take 4 coins ($4 \times 8¢$).

[1]Because 1 is a valid coin, there will always be a coin to choose from

Problem #7.2

Problem Statement. *Suppose we have an unlimited number of rooms and a finite number of activities, each of which can be staged in any of the rooms. Give an efficient algorithm that can schedule all the activities using the smallest number of rooms.*

A greedy algorithm that always produces solution is as follows:

1. Sort all of the activities in respect to their finish times (i.e., the job that finishes first is the first element). Call this sorted set of activities A.

2. Take a particular room that is not preoccupied R_i and map a particular schedule S_i to it.

3. While not at the end of A:

   a) Pick the first activity as just the first element in the A. Remove this element from A and add it the schedule S_i.

   b) Search A from beginning, finding the first activity who's start time is after the finish time of the previous job. Add this to schedule S_i, remove it from A.

   c) Repeat the last step.

4. If A is empty, the algorithm is finished. If not, repeat Step 2.

Question #8

Problem #8.1

Problem Statement. *Consider the following problem: given a graph, determine whether it can be colored using exactly* 4 *colors. Prove that this coloring problem is NP-Complete.*

Proof. To prove the Four Coloring Problem is NP-Complete, first we must prove that the Four Color Problem is NP. To check a solution in polynomial time, we simple iterate through all edges, and check:

- Make sure that the graph is colored by ≤ 4 colors.
- Iterate though all edges, ensuring that every edge's neighbor is colored by a different color.

Because this is a $\mathcal{O}\left({}_num_1\right)$ algorithm, we can check a solution in polynomial time.

To prove that this is NP-Complete, we will reduce it to the three coloring problem, as follows.

Supposing we have a graph G, we wish to create a new graph G', such that if G is colorable in three colors, G' can be colored in four. To make G', then we simply add a new vertex and attach it to all of the verticies in G. Therefore, if G is three colorable, then we know G' to be four colorable.

Because we know the Four Coloring Problem is NP and is reducable from the Three Coloring Problem, we conclude that the Four Coloring Problem is NP-Complete. □

Problem #8.2

Problem Statement. *Prove that for all $k > 4$, determining whether a graph can be colored using exactly k colors is NP-Complete.*

Proof. To prove the Four Coloring Problem is NP-Complete, first we must prove that the k-Color Problem is NP. To check a solution in polynomial time, we simple iterate through all edges, and check:

- Make sure that the graph is colored by $\leq k$ colors.
- Iterate though all edges, ensuring that every edge's neighbor is colored by a different color.

Because this is a $\mathcal{O}\left({}_n um_1\right)$ algorithm, we can check a solution in polynomial time.

To prove that this is NP-Complete, we will reduce it to the three coloring problem, as follows.

Supposing we have a graph G, we wish to create a new graph G', such that if G is colorable in three colors, G' can be colored in k. To make G', then we simply add attach k new vertices and attach it to all of the vertices in G. Therefore, if G is three colorable, then we know G' to be k colorable.

Because we know the k-Coloring Problem is NP and is reducable from the Three Coloring Problem, we conclude that the k-Coloring Problem is NP-Complete.

□

CS5402

Data Mining

Test #1

CS5402 — Intro To Data Mining

Illya Starikov

Due Date: July 15th, 2018

Multiple Choice

1. **e. None of the above**
2. **c. Remove any attribute that has missing values.**
3. **b.** $\frac{1}{2}$
4. **b. wt**
5. **d. Spearman's rank correlation coefficient**
6. **c. Healthland**
7. **b. slice for Time = Q1**
8. **d. roll up on Location = Beijing or Tokyo (i.e., from city to country)**
9. **c. drill down on Time = Q1 (i.e., from quarter to month)**
10. **a. dice for (location = Beijing or Tokyo) and (product = Chain or bracelet) and (time = Q1 or Q2)**

11 Short Answer

Method #1 is the most accurate, because the true positive (y-axis) correctly identified the values, while the false positive (x-axis) incorrectly identified the values. Method #1 had the fastest growing function (with respect to y).

12 1-R Method

Attribute	**Attribute Value**	**# Rows With Attribute Value**	**Most Frequent Value For sportPref**	**Errors**	**Total Errors**
ageGroup	youngAdult	3	football (2)	1	3
	middleAge	3	football/hockey/baseball (1/1/1)	2	
	senior	2	baseball (2)	0	
gender	M	5	baseball/football (2/2)	3	5
	F	3	football/hockey/baseball (1/1/1)	2	
petPreference	dog	5	football (3)	2	3
	cat	3	baseball (2)	1	

The rules are as follows:

$$\begin{aligned} \text{ageGroup} = \textbf{youngAdult} &\implies \text{football} \\ \text{ageGroup} = \textbf{middleAge} &\implies \text{football} \\ \text{ageGroup} = \textbf{senior} &\implies \text{baseball} \end{aligned}$$

13 Prism

For football, we get the following table:

gender	pet	drink	sport
M	dog	beer	football
F	dog	beer	football

For our P and T values:

	T	P	T/P
gender = M	3	1	1/3
gender = F	4	1	1/4
pet = dog	3	2	2/3
drink = beer	3	2	3/4

Seeing as not T/P values are 1, we must add a clause. We choose pet = dog as the base.

	T	P	T/P
gender = M	1	0	0
gender = F	1	0	0
drink = beer	2	2	1

$$\text{pet} = \textbf{dog} \text{ and drink} = \textbf{beer} \implies \text{football}$$

14 Statistical Modeling

The likelihood would be as follows:

$$\text{likelihood} = {}^{4}\!/_{9} \times {}^{2}\!/_{9} \times {}^{6}\!/_{9} \times {}^{3}\!/_{9} \times {}^{9}\!/_{14}$$

15 Entropy

(a) entropyBeforeSplit would be as follows:

$$-{}^{1}\!/_{6}\,\log_2\left({}^{1}\!/_{6}\right) - {}^{2}\!/_{6}\,\log_2\left({}^{2}\!/_{6}\right) - {}^{3}\!/_{6}\,\log_2\left({}^{3}\!/_{6}\right)$$

(b) entropyPoor would be as follows:

$$-{}^{2}\!/_{4}\,\log_2\left({}^{2}\!/_{4}\right) - {}^{2}\!/_{4}\,\log_2\left({}^{2}\!/_{4}\right)$$

(c) infoGain would be determined as follows:

$$\begin{aligned}\text{entropyAfterSplit} &= {}^{3}\!/_{6}\,\text{entropyShort} + {}^{2}\!/_{6}\,\text{entropyMed} + {}^{1}\!/_{6}\,\text{entropyLong}\\ \text{infoGain} &= \text{entropyBeforeSplit} - \text{entropyAfterSplit}\end{aligned}$$

16 Rule Induction

(a) The partitions would be as follows:

$$\begin{aligned}\{d\}^* &= \{\{x_1\},\{x_2,x_3\},\{x_5\},\{x_5\}\}\\ \{e\}^* &= \{\{x_1,x_2,x_5\},\{x_3,x_4\}\}\\ \{d,\ e\}^* &= \{\{x_1\},\{x_2\},\{x_3\},\{x_4\},\{x_5\}\}\end{aligned}$$

(b) The coverings are as follows:

- $\{d\}^*$ would not work, because every block in the partition is not a subset of a block in $\{f\}^*$.
- $\{d,e\}^*$ would work, because every block in the partition is a subset of a block in $\{f\}^*$.
- $\{a,d,e\}^*$ would not work, because although every block in the partition is a subset of a block in $\{f\}^*$, it is not minimal.

(c) The rules would be as follows:

$$\begin{aligned}
d = \text{X and } e = 4 &\implies f = T \\
d = \text{S and } e = 4 &\implies f = T \\
d = \text{S and } e = 3 &\implies f = F \\
d = \text{H and } e = 3 &\implies f = F \\
d = \text{M and } e = 4 &\implies f = F
\end{aligned}$$

17 KD-Tree

Sorting, we get the following: [(2, 10), (4, 20), (6, 10), (8, 20), (10, 30)].
With a median of 6...

- $x < 6$ group: [(2, 10), (4, 20)]
- $x \geq 6$ group: [(6, 10), (8, 20), (10, 30)]

Sorting, we get the following: [(2, 10), (4, 20)] [(6, 10), (8, 20), (10, 30)]
With a median of 15 for the first group:

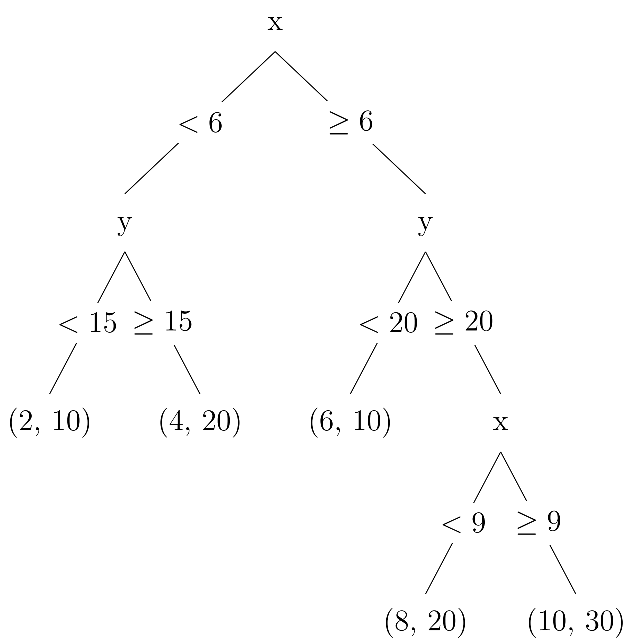
- $y < 15$ group: [(2, 10)]
- $y \geq 15$ group: [(4, 20)]

With a median of 20 for the second group:

- $y < 20$ group: (6, 10)
- $y \geq 20$ group: [(8, 20), (10, 30)]

(Using a shortcut for the final block), Sorting, and using a median of 9, our last block looks like as follows:

- $x < 9$ group: [(8, 20)]
- $y \geq 9$ group: [(10, 30)]



18 Clustering

x	y	distance to (2, 4)	distance to (5, 6)	distance to (8, 1)
2	4	0	5	9
5	6	5	0	8
8	1	9	8	0
7	3	6	5	3
4	10	8	5	13
3	0	5	8	6
9	8	11	6	8

Our clusters would be as follows:

Cluster Center (2, 4) (2, 4), (3, 0)

Cluster Center (5, 6) (5, 6), (4, 10), (9, 8)

Cluster Center (8, 1) (8, 1), (7, 3)

With means as follows:

Cluster Mean of (2, 4), (3, 0) $(2.5, 2) \approx (3, 2)$

Cluster Mean of (5, 6), (4, 10), (9, 8) (6, 8)

Cluster Center of (8, 1), (7, 3) (7.5, 2) ≈ (8, 2)

x	y	distance to (3, 2)	distance to (6, 8)	distance to (8, 2)
2	4	3	8	8
5	6	6	3	7
8	1	6	9	1
7	3	5	6	2
4	10	9	4	12
3	0	2	11	7
9	8	12	3	7

Cluster Center (3, 2) (2, 4), (3, 0)

Cluster Center (6, 8) (5, 6), (4, 10), (9, 8)

Cluster Center (8, 2) (8, 1), (7, 3)

Clusters haven't changed! Final cluster centers and instances are as follows:

Cluster Center (3, 2) (2, 4, 11, yes), (3, 0, 3, yes)

Cluster Center (6, 8) (5, 6, 5, no), (4, 10, 8, yes), (9, 8, 1, no)

Cluster Center (8, 2) (8, 1, 7, no), (7, 3, 4, yes)

19 Confusion Table

(a) For a randomly produced results, there were 8 values that we predicted to be B, when they were actually G.

(b) For a classifier produced results, there were 30 values that we predicted to be B, and were actually B.

(c) The non-random classifier, 90 were predicted correctly. For the random classifier, 39 were predicted correctly. Therefore, 51 more were predicted correctly.

(d) Kappa Statistic would be

$$\frac{\text{Non-Random Correct - Random Correct}}{\text{Total}}$$

Which would be as follows:

$$\frac{90 - 39}{100}$$

Test #2

CS5402 — Intro To Data Mining

Illya Starikov

Due Date: July 29th, 2018

1 C4.5

Split = 120

$$\begin{aligned} \text{EntropyLT} &= -{}^{3}/_{4} \log_2 ({}^{3}/_{4}) - {}^{1}/_{4} \log_2 ({}^{1}/_{4}) \\ \text{EntropyGt} &= -{}^{2}/_{5} \log_2 ({}^{2}/_{5}) - {}^{3}/_{5} \log_2 ({}^{3}/_{5}) \\ \text{Info Gain} &= X - ({}^{4}/_{9} \times \text{EntropyLT} + {}^{5}/_{9} \times \text{EntropyGT}) \end{aligned}$$

Split = 140

$$\begin{aligned} \text{EntropyLT} &= -{}^{4}/_{6} \log_2 ({}^{4}/_{6}) - {}^{2}/_{6} \log_2 ({}^{2}/_{6}) \\ \text{EntropyGt} &= -{}^{2}/_{3} \log_2 ({}^{2}/_{3}) - {}^{1}/_{3} \log_2 ({}^{1}/_{3}) \\ \text{Info Gain} &= X - ({}^{6}/_{9} \times \text{EntropyLT} + {}^{3}/_{9} \times \text{EntropyGT}) \end{aligned}$$

We choose the split that we calculated previously based on it's associated **highest information gain**. Values before it should be assigned *Less Than Split* and values after it should be assigned *Greater Than Split*.

2 Grouping Or Splitting

{{SciFiction, Mystery}, {NonFiction}}

$$\begin{aligned} \text{SciMystery} &= -{}^{3}/_{6} \log_2 ({}^{3}/_{6}) - {}^{2}/_{6} \log_2 ({}^{2}/_{6}) - {}^{1}/_{6} \log_2 ({}^{1}/_{6}) \\ \text{NonFiction} &= -0 - {}^{1}/_{6} \log_2 ({}^{1}/_{6}) - {}^{1}/_{6} \log_2 ({}^{1}/_{6}) \\ \text{Info Gain} &= 1.5575 - ({}^{6}/_{8} \times \text{SciMystery} + {}^{2}/_{8} \times \text{NonFiction}) \end{aligned}$$

{{SciFiction, NonFiction}, {Mystery}}

$$\begin{aligned} \text{SciNonFiction} &= -{}^{2}/_{4} \log_2 ({}^{2}/_{4}) - {}^{1}/_{4} \log_2 ({}^{1}/_{4}) - {}^{1}/_{4} \log_2 ({}^{1}/_{4}) \\ \text{NonFiction} &= -{}^{1}/_{4} \log_2 ({}^{1}/_{4}) - {}^{2}/_{4} \log_2 ({}^{2}/_{4}) - {}^{1}/_{4} \log_2 ({}^{1}/_{4}) \\ \text{Info Gain} &= 1.5575 - ({}^{4}/_{8} \times \text{SciNonFiction} + {}^{4}/_{8} \times \text{Mystery}) \end{aligned}$$

$$\{\{Mystery,\ NonFiction\},\ \{NonFiction\}\}$$

$$\begin{aligned}\text{SciNonFiction} &= -{}^{1}\!/_{6}\log_2({}^{1}\!/_{6}) - {}^{3}\!/_{6}\log_2({}^{3}\!/_{6}) - {}^{2}\!/_{6}\log_2({}^{2}\!/_{6})\\ \text{NonFiction} &= -{}^{2}\!/_{2}\log_2({}^{2}\!/_{2}) - 0 - 0\\ \text{Info Gain} &= 1.5575 - ({}^{6}\!/_{8}\times\text{SciNonFiction} + {}^{2}\!/_{8}\times\text{NonFiction})\end{aligned}$$

To determine if any attributes are to be grouped, take the **highest information gain**, and compare it to the Entropy Before Split for music preference. If information gain is higher, then the grouping was better; if not, the grouping was worse.

3 Support Vectors

The equations would reduce down to:

$$\begin{aligned}10\,\alpha_1 + 10\,\alpha_2 &= -1\\ 10\,\alpha_1 + 11\,\alpha_2 &= 1\end{aligned}$$

For which we get the following solutions:

$$\alpha_1 = -\frac{21}{10} \qquad \alpha_2 = 2$$

For the discriminating 2D hyperplane, we get as follows:

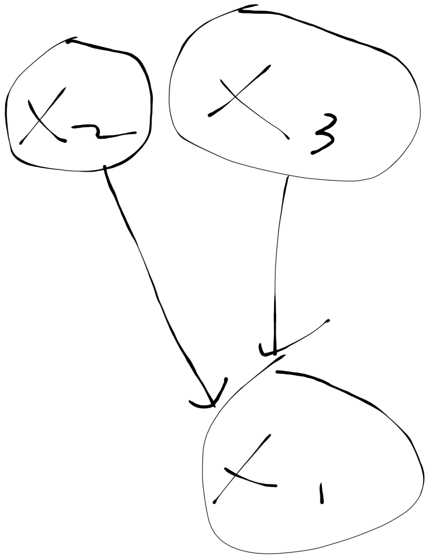
$$\begin{aligned}w &= -\frac{21}{10}\times s_1 + 2\times s_2\\ &=< -\frac{3}{10}, 2, -\frac{41}{10} >\end{aligned}$$

Meaning our equation would be as follows:

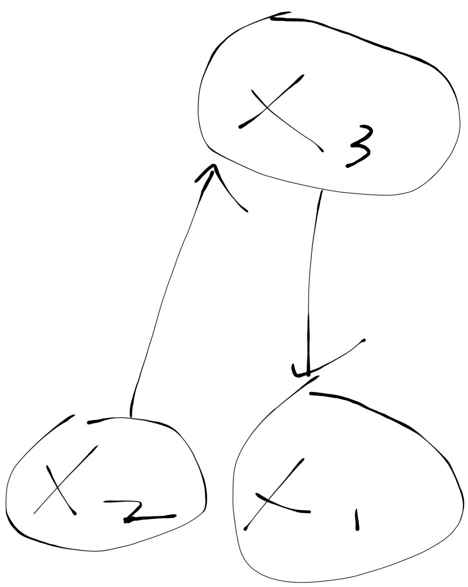
$$-{}^{3}\!/_{10}\,x - 2\,y - {}^{41}\!/_{10} = 0$$

4 Bayesian Network

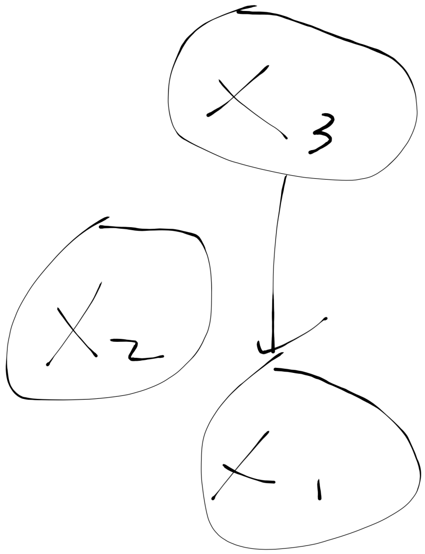
The algorithm will enumerate the possible options (in the original ordering) for a parent of X2. It does so by calculating the probability of the particular attribute being a parent of X2, and the one with the greatest probability is the node that becomes the parents.



1.



2.



3.

5 DBScan

	density	designation
A1	2	Border
A2	2	Border
A3	3	Core
A4	1	Noise
A5	2	Border
A6	2	Border
A7	2	Noise
A8	1	Noise
A9	2	Noise
A10	3	Core

The clusters would be as follows: $\{A1, A2, A3\}$ and $\{A5, A6, A10\}$. Points with have are dense (density > number of points) are designated as core points. Points that are around a core points (within an ϵ of a core point) are designated as border point. Points that fall in neither of these categories are designated as noise.

6 Linkage

The single linkage would be as follows:

$$|(1,\ 4) - (3,\ 5)| = 3$$

The complete linkage would be as follows:

$$|(1,\ 2) - (4,\ 5)| = 6$$

The centroid linkage would be as follows:

$$|(1,\ {}^{11}\!/\!_{4}) - ({}^{14}\!/\!_{4},\ 5)| = {}^{19}\!/\!_{4}$$

The average linkage would be as follows:

$$\text{Average of all distances between points} = 5$$

7 Ensemble Classifier

Sum	-2	4	-6	2
Class	-1	1	-1	1

8 Bayes Network

For **worker = T**, the likelihood would be as follows:

$$0.8 \times 0.02 \times 0.1 \times 0.1 \times 0.01$$

For **worker = F**, the likelihood would be as follows:

$$0.8 \times 0.02 \times 0.1 \times 0.1 \times 0.99$$

The probability for **worker = T** would be:

$$\frac{0.8 \times 0.02 \times 0.1 \times 0.1 \times 0.01}{0.8 \times 0.02 \times 0.1 \times 0.1 \times 0.01 + 0.8 \times 0.02 \times 0.1 \times 0.1 \times 0.99}$$

And the probability for **worker = F** would be:

$$\frac{0.8 \times 0.02 \times 0.1 \times 0.1 \times 0.99}{0.8 \times 0.02 \times 0.1 \times 0.1 \times 0.01 + 0.8 \times 0.02 \times 0.1 \times 0.1 \times 0.99}$$

Multiple Choice

9. d. repeatedly sampling from the original dataset according to a uniform probability distribution

10. c. increased, decreased

11. a. bias, variance
12. b. false
13. c. 180 instances reached this point in the decision tree, but 22 of those were not classified as tested_negative in the training dataset
14. d. age > 34
15. b. compute the predicted error rate of the rule with one condition (C1, C2, C3) deleted and no conditions deleted, and, from those, use the version of the rule with the lowest rate
16. c. each instance in the dataset will have a probability of being in a particular cluster
17. b. supervised

MATH1214

Calculus I

Calculus I: Single-Variable Calculus

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August 6, 2025

Contents

num_1!

0 Functions

0.1 Review of Functions

A **function** is a rule that assigns to each value x in a set D a unique value denoted $f(x)$. The set D is the **domain** of the function. The **range** is the set of all values of $f(x)$ produced as x varies over the domain.

Vertical Line Test

A graph represents a function if and only if it passes the **vertical line test**. Every vertical line intersects the graph at most once. A graph that fails this test does not represent a function.

Composite Functions

Given two functions f and g, the composite functions $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$. It is evaluated in two steps: $y = f(u)$, where $u = g(x)$. The domain of $f \circ g$ consists of all x in the domain of g such that $u = g(x)$ is in the domain of f.

Symmetry in Functions

An **even function** has the property that $f(-x) = f(x)$, for all x in the domain. The graph of an even function symmetric about the y-axis. Polynomials consisting of only even powers of the variable (of the form x^{2n}, where n is a nonnegative integer) are even functions.
An **odd function** f has the property that $f(-x) = -f(x)$, for all x in the domain. The graph of an odd function is symmetric about the origin. Polynomials consisting of only odd powers of the variable (of the form x^{2n+1}, where n is a nonnegative integer) are odd functions.

0.2 Review of Functions

A **function** is a rule that assigns to each value x in a set D a unique value denoted $f(x)$. The set D is the **domain** of the function. The **range** is the set of all values of $f(x)$ produced as x varies over the domain.

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An **odd function** f has the property that $f(-x) = -f(x)$, for all x in the domain. The graph of an odd function is symmetric about the origin. Polynomials consisting of only odd powers of the variable (of the form x^{2n+1}, where n is a nonnegative integer) are odd functions.

0.3 Representing Functions

Some brief families of functions can include

Polynomials are functions of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where the **coefficients** $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$ and the nonnegative integer n is the **degree** of the polynomial. The domain of any polynomial is the set of all real numbers. An nth-degree polynomial can have as many as n real **zeros** or **roots** — values of x.

Rational Functions are ratios of the form $f(x) = p(x)/q(x)$, where p and q are polynomials. Because division by zero is prohibited, the domain of a rational function is the set of all real numbers except those for which the denominator is zero.

Algebriac Functions are constructed using the operations of algebra: addition, subtraction, multiplication, division, and roots. Examples of algebraic functions are $f(x) = \sqrt{2x^3 + 4}$ and $f(x) = x^{1/4}(x^3 + 2)$. In general, if an even root (square root, fourth root, and so forth) appears, then the domain does not contain points at which the quantity under the root is negative (and perhaps other points).

Exponential Functions have the form $f(x) = b^x$, where the base $b \neq 1$ is a positive real number. Closely associated with exponential functions are logarithmic functions of the form $f(x) = \log_b x$, where $b > 0$ and $b \neq q$. An exponential function has a domain consisting of all real numbers. Logarithmic functions are defined for positive, real numbers. The most important function is the **natural exponential function** $f(x) = e^x$, with base $b = e$, where $e \approx 2.71828\ldots$ is one of the fundamental constants of mathematics. Associated with the natural exponential function is the **natural logarithmic function** $f(x) = \ln x$, which also has the base $b = e$.

Trigonometric Functions are $\sin x$, $\cos x$, $\tan x$, $\sec x$, and $\cos x$; they are fundamental to mathematics and many areas of application. Also important are their relatives, the **inverse trigonometric functions**.

Transcendental Functions Trigonometric, exponential, and logarithmic functions are few examples of a large family called transcendental functions.

Transformations

Given the real numbers a, b, c, and d and the function f, the graph of $y = cf(a(x - b)) + d$ is obtained from the graph of $y = f(x)$ in the following steps.

$$
\begin{aligned}
y = f(x) &\xrightarrow{\text{horizontal scaling by a factor of } |a|} y = f(ax) \\
&\xrightarrow{\text{horizontal shift by } b \text{ units}} y = f(a(x-b)) \\
&\xrightarrow{\text{vertical scaling by a factor of } |c|} y = cf(a(x-b)) \\
&\xrightarrow{\text{horizontal scaling by a factor of |a|}} y = cf(a(x-b)) + d
\end{aligned}
$$

0.4 Inverse, Exponential, and Logarithmic Functions

The Natural Exponential Function

The **natural exponential function** is $f(x) = e^x$, which as the base $e = 2.718281828459\ldots$.

Inverse Function

Given a function f, its inverse (if it exists) is a function f^{-1} such that whenever $y = f(x)$, then $f^{-1}(y) = x$.

One-to-One Functions and the Horizontal Line Test

A function f is **one-to-one** on a domain D if each value of $f(x)$ corresponds to exactly one value of x in D. More precisely, f is one-to-one on D is $y(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ for x_1 and x_2 in D. The **horizontal line test** says that every horizontal line intersects the graph of a one-to-one function at most once.

Existence of Inverse Functions

Let f be one-to-one function on a domain D with a range R. Then f has a unique inverse f^{-1} with domain R and range D such that

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(y)) = y,$$

where x is in D and y is in R.

Finding an Inverse Function

Suppose f is one-to-one on an interval I. To find f^{-1}:

- Solve $y = f(x)$ for x. If necessary, choose the function that corresponds to I.
- Interchange x and y and write $y = f^{-1}(x)$.

Logarithmic Function Base b

For any base $b > 0$, with $b \neq 1$, the **logarithmic function base** b, denoted $y = \log_b x$, is the inverse of the exponential function $y = b^x$. The inverse of the natural exponential function with base $b = e$ is the **natural logarithm function**, denoted $y = \ln x$.

Inverse Relations For Exponential and Logarithmic Functions

For any base $b > 0$, with $b \neq 1$, the following inverse relations hold:

- $b^{\log_b x} = x$, for $x > 0$
- $\log_b b^x = x$, for any real values of x

Change-of-Base Rules

Let b be a positive real number with $b \neq 1$. Then

$$b^x = e^{x \ln b}, \text{for all } x \qquad \text{and} \qquad \log_b x = \frac{\ln x}{\ln b}, \text{for } x > 0$$

More generally, if c is a positive real number with $c \neq 1$, then

$$b^x = c^{x \log_c b}, \text{for all } x \qquad \text{and} \qquad \log_b x = \frac{\log_c x}{\log_c b}, \text{for } x > 0$$

0.5 Trigonometric Functions and Their Inverses

Let $P(x, y)$ be a point on a circle of radius r associated with the angle θ. Then

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \tag{1}$$

$$\cot\theta = \frac{x}{y} \qquad \sec\theta = \frac{r}{x} \qquad \csc\theta = \frac{r}{y} \tag{2}$$

$$\tag{3}$$

Trigonometric Identities

Reciprocal Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} \tag{4}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \tag{5}$$

Pythagorean

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \cot^2\theta = \csc^2\theta \qquad \tan^2\theta + 1 = \sec^2\theta \tag{6}$$

Double- and Half-Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \sin^2\theta - \sin^2\theta \tag{7}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2} \tag{8}$$

Period of Trigonometric Function

The function $\sin\theta$, $\cos\theta$, $\sec\theta$, and $\csc\theta$ have a period of 2π

$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta \tag{9}$$

$$\sec(\theta + 2\pi) = \sec\theta \qquad \csc(\theta + 2\pi) = \csc\theta \tag{10}$$

for all θ in the domain.
The functions $\tan\theta$ and $\cot\theta$ have a period of π:

$$\tan(\theta + \pi) = \tan\theta \qquad \cot(\theta + \pi) = \cot\theta \tag{11}$$

for all θ in the domain.

Inverse Sine and Cosine

$y = \sin^{-1} x$ is the value of y such that $x = \sin y$, where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. $y = \cos^{-1} x$ is the value of y such that $x = \cos y$, where $0 \le y \le \pi$. The domain of both $\sin^{-1} x$ and $\cos^{-1} x$ is $\{x : -1 \le x \le 1\}$.

Other Inverse Trigonometric Functions

- $\tan^{-1} x$ is the value of y such that $x = \tan y$, where $-\frac{\pi}{2} < \frac{\pi}{2}$.
- $\cot^{-1} x$ is the value of y such that $x = \tan y$, where $0 < y < \pi$.

The domain of both $\tan^{-1} x$ and $\cot^{-1} x$ is $\{x : -\infty < x < \infty\}$

- $\sec^{-1} x$ is the value of y such that $x = \sec y$, where $0 < y < \pi$ with $y \neq \frac{\pi}{2}$
- $\tan^{-1} x$ is the value of y such that $x = \tan y$, where $-\frac{\pi}{2} < \frac{\pi}{2}$.

1 Limits

1.2 Definitions of Limits

Limits of a Function (Preliminary)

Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = L$$

and say the limit of $f(x)$ as x approaches a equals L.

One-Sided Limits

1 Right-sided limits Suppose f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write

$$\lim_{x \to a^+} f(x) = L \qquad (12)$$

and say the limit of $f(x)$ as x approaches a from the right equals L.

2 Left-sided limits Suppose f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write

$$\lim_{x \to a^-} f(x) = L \tag{13}$$

and say the limit of $f(x)$ as x approaches a from the left equals L.

Relationship Between One-Sided and Two-Sided Limits

Assume f is defined for all x near a except possibly at a. Then $\lim_{{}_{n}um_1} f(x) = L$ if and only if $\lim_{{}_{n}um_1} = \lim_{{}_{n}um_1} = L$.

1.3 Techniques For Computing Limits

Limits of Linear Functions

Let a, b, and m be real numbers. For Linear functions $f(x) = mx + b$,

$$\lim_{x \to a} f(x) = f(a) = ma + b \tag{14}$$

Limit Laws

Assume $\lim_{{}_{n}um_1} f(x)$ and $\lim_{{}_{n}um_1} g(x)$ exist. The following properties hold, where c is a real number, and $m > 0$ and $n > 0$ are integers.

1 Sum $\lim_{{}_{n}um_1} [f(x) + g(x)] = \lim_{{}_{n}um_1} f(x) + \lim_{{}_{n}um_1} g(x)$

2 Difference $\lim_{{}_{n}um_1} [f(x) - g(x)] = \lim_{{}_{n}um_1} f(x) - \lim_{{}_{n}um_1} g(x)$

3 Constant Multiple $\lim_{{}_{n}um_1} [c\,f(x)] = c \lim_{{}_{n}um_1}$

4 Product $\lim_{{}_{n}um_1} [f(x)g(x)] = \left[\lim_{{}_{n}um_1} f(x)\right] \left[\lim_{{}_{n}um_1} g(x)\right]$

5 Quotient $\lim_{{}_{n}um_1} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{{}_{n}um_1} f(x)}{\lim_{{}_{n}um_1} g(x)}$, provided $\lim_{{}_{n}um_1} g(x) \neq 0$

6 Power $\lim_{{}_{n}um_1} [f(x)]^n = \left[\lim_{{}_{n}um_1} f(x)\right]^n$

7 Fractional Power $\lim\limits_{num_1} [f(x)]^{n/m} = \left[\lim\limits_{num_1} f(x)\right]^{n/m}$, provided $f(x) \geq 0$, for x near a, if m is even and n/m is reduced to lowest terms.

Limits of Polynomial and Rational Functions

Assume p and q are polynomials and a is a constant

- Polynomial functions: $\lim\limits_{num_1} p(x) = p(a)$
- Rational functions: $\lim\limits_{num_1} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$, provided $q(a) \neq 0$

Limit Laws For One-Sided Limits

Laws 1–6 hold with $\lim\limits_{num_1}$ replaced by $\lim\limits_{num_1}$ or $\lim\limits_{num_1}$. Law 7 is modified as follows, assume $m > 0$ and $n > 0$ are integers.

7 Fractional Power

- $\lim\limits_{num_1} [f(x)]^{n/m}$, provided $f(x) \geq 0$, for x near a with $x > a$, if m is even and n/m is reduced to lowest terms
- $\lim\limits_{num_1} [f(x)]^{n/m}$, provided $f(x) \geq 0$, for x near a with $x < a$, if m is even and n/m is reduced to lowest terms

The Squeeze Theorem

Assume the function f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$, for all values of x near a, except possibly at a. If $\lim\limits_{num_1} f(x) = \lim\limits_{num_1} h(x) = L$, then $\lim\limits_{num_1} g(x) = L$.

1.4 Infinite Limits

Suppose f is defined for all x near a. If $f(x)$ grows arbitrarily large for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = \infty \tag{15}$$

We say the limit of $f(x)$ as x approaches a is infinity.
If $f(x)$ is negative and grows arbitrarily large in magnitude for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = -\infty \tag{16}$$

In this case, we say the limit of $f(x)$ as x approaches a is negative infinity. In both cases, *the limit does not exist.*

One-Sided Infinite Limits

Suppose f is defined for all x near a with $x > a$. If $f(x)$ becomes arbitrarily large for all x sufficiently close to a with $x > a$, we write $\lim_{num_1} f(x) = \infty$. The one-sided infinite limit $\lim_{num_1} = -\infty$, $\lim_{num_1} f(x) = \infty$, and $\lim_{num_1} = -\infty$ are defined analogously.

Vertical Asymptote

If $\lim_{num_1} f(x) = \pm\infty$, $\lim_{num_1} = \pm\infty$ or $\lim_{num_1} f(x) = \pm\infty$, the line $x = a$ is called a **vertical asymptote** of f.

1.5 Limits at Infinity

If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write

$$\lim_{x \to \infty} f(x) = L \tag{17}$$

We say the limit of $f(x)$ as x approaches infinity is L. In this case the line $y = L$ is a **horizontal asymptote** of f. The limit at negative infinity, $\lim_{num_1} f(x) = M$, is defined analogously. When the limit exists, the horizontal asymptote is $y = M$.

Infinite Limits at Infinity

If $f(x)$ becomes arbitrarily large as x becomes arbitrary large, then we write

$$\lim_{x \to \infty} f(x) = \infty \tag{18}$$

The limits $\lim_{num_1} = -\infty$, $\lim_{num_1} = \infty$, and $\lim_{num_1} = -\infty$ are defined similarly.

Limit of Infinity at Powers and Polynomials

Let n be a positive integer and let p be the polynomial $p(x) = a_n x^m + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, where $a_n \neq 0$.

1. $\lim_{num_1} x^n = \infty$ when n is even.

2. $\lim_{num_1} x^n = \infty$ and $\lim_{num_1} x^n = -\infty$ when n is odd.

3. $\lim_{num_1} \frac{1}{x^n} = \lim_{num_1} x^{-n} = 0$

4. $\lim_{num_1} p(x) = \infty or - \infty$, depending on the degree of the polynomial or the leading coefficient of a_n.

End Behavior and Asymptotes of Rational Functions

Suppose $f(x) = \frac{p(x)}{q(x)}$ is a rational function, where

$$p(x) = a_m x^m + a_{m-1} + \cdots + a_2x^2 + a_1x + a_0 \tag{19}$$

and

$$q(x) = b_m x^m + b_{m-1} + \cdots + b_2x^2 + b_1x + b_0 \tag{20}$$

with $a_m \neq 0$ and $b_n \neq 0$.

1. If $m < n$ then $\lim_{num_1} f(x) = 0$, and $y = 0$ is a horizontal asymptote of f.

2. If $m = n$, then $\lim_{num_1} f(x) = a_m/b_n$, and $y = a_m/b_n$ is a horizontal asymptote.

3. If $m > n$, then $\lim_{num_1} f(x) = \infty$ or $-\infty$, and f has no horizontal asymptote.

4. If $m = n+1$, then $\lim_{num_1} f(x) = \infty$ or $-\infty$, f has no horizontal asymptote, but f has a slant asymptote.

5. Assuming that $f(x)$ is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeros of q.

End Behavior of e^x, e^{-x}, and $\ln x$

The end behavior for e^x and e^{-x} on $(-\infty, \infty)$ and $\ln x$ on $(0, \infty)$ is given by the following limits:

$$\lim_{x\to\infty} e^x = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} e^x = 0 \tag{21}$$

$$\lim_{x\to\infty} e^{-x} = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} e^x = 0 \tag{22}$$

$$\lim_{x\to 0^+} \ln x = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} \ln x = \infty \tag{23}$$

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Calculus II

Chapter 6: Applications of Integration

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6.7 Physical Applications

Mass of a One-Dimensional Object

Suppose a thin bar or wire is represented by a line segment on the interval $a \le x \le b$ with a density function ρ (with units of mass per length). The mass of the object is

$$m = \int_a^b \rho(x)\, dx \tag{1}$$

Work

The work done by a variable force F in moving an object along a line from $x = a$ to $x = b$ in the direction of the force is

$$W = \int_a^b F(x)\, dx \tag{2}$$

Solving Lifting Problems

1. Draw a y-axis in the vertical direction (parallel to gravity) and choose a convenient origin. Assume the interval $[a, b]$ corresponds to the vertical extent of the fluid.

2. For $a \le y \le b$, find the cross-sectional area $A(y)$ of the horizontal slices and the distance $D(y)$ the slices must be lifted.

3. The work required to lift the water is

$$W = \int_a^b \rho g A(y) D(y)\, dy \tag{3}$$

Solving Force/Pressure Problems

1. Draw a y-axis on the face of the dam in the vertical direction and choose a convenient origin (often taken to be the base of the dam).

2. Find the width function $w(y)$ for each value of y on the face of the dam.

3. If the base of the dam is at $y = 0$ and the top of the dam is at $y = a$, then the total force on the dam is

$$F = \int \rho g(a - y)w(y)\, dy \tag{4}$$

Chapter 7: Logarithmic and Exponential Functions

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7.1 Inverse Function

Derivative of the Inverse Function

Let f be differentiable and have an inverse on an interval I. If x_0 is a point of I at which $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}, \text{ where } y_0 = f(x_0) \tag{1}$$

7.2 The Natural Logarithmic and Exponential Functions

The Natural Logarithm

The **natural logarithm** of a number $x > 0$, denoted x, is defined

$$\ln x = \int_1^x \frac{dt}{t} \tag{2}$$

Properties of the Natural Logarithm

1. The domain and range of $\ln x$ are $(0, \infty)$ and $(-\infty, \infty)$, respectively.
2. $\ln(xy) = \ln x + \ln y, \forall x, y \in \mathbf{R}^+$
3. $\ln(x/y) = \ln x - \ln y, \forall x, y \in \mathbf{R}^+$
4. $\ln x^p = p \ln x, \forall x \in \mathbf{Q}^+$
5. $\frac{d}{dx}(\ln |x|) = \frac{1}{x}, \forall x \in \mathbf{R} \wedge x \neq 0$
6. $\frac{d}{dx}(\ln |u(x)|) = \frac{u'(x)}{u(x)}$
7. $\int \frac{dx}{x} = \ln |x| + C$

The Number e

The number e is the real number that satisfies

$$\ln e = \int_1^e \frac{dt}{t} = 1 \tag{3}$$

The Exponential Function

$\forall x, y \in \mathbf{R}$

$$y = e^x \Leftrightarrow x = \ln y \tag{4}$$

Properties of e

- $e^{x+y} = e^x e^y$
- $e^{x-y} = e^x / e^y$
- $(e^x)^y = e^{xy}, \forall y \in \mathbf{Q}$
- $\ln(e^x) = x, \forall x \in \mathbf{R}$
- $e^{\ln\ x} = x, \forall x \in \mathbf{R}^+$

Exponential Functions with General Bases

Let $b \in \mathbf{R}^+ \wedge b \neq 1$. $\forall x \in \mathbf{R}^+$,

$$b^x = e^{x \ln b} \tag{5}$$

Derivative and Integral of the Exponential Function

$\forall x \in \mathbf{R}$,

$$\frac{d}{dx}(e^{u(x)}) = e^{u(x)} u'(x) \tag{6}$$

$$\int e^x \, dx = e^x + C \tag{7}$$

7.3 Logarithmic and Exponential Functions with Other Bases

Logarithmic Function Base b

For any base $b > 0$, with $b \neq 1$, the **logarithmic function base** b, denoted $\log_b x$, is the inverse of the exponential function b^x.

Inverse Relations for Exponential and Logarithmic Functions

For any base $b > 0$, with $b \neq 1$, the following inverse relation holds.

- $b^{\log_b x} = x$, $\forall x \in \mathbf{R}^+$
- $\log_b b^x = x$, $\forall x$

Derivative of b^x

If $b > 0 \wedge b \neq 1, \forall x$,

$$\frac{d}{dx}(b^x) = b^x \ln b \tag{8}$$

Indefinite Integral of b^x

For $b > 0 \wedge b \neq 1$,

$$\int b^x dx = \frac{1}{\ln b} b^x + C \tag{9}$$

General Power Rule

$\forall p, x \in \mathbf{R}^+$,

$$\frac{d}{dx}(x^p) = px^{p-1} \tag{10}$$

Furthermore, if u is a positive differentiable function on its domain, then

$$\frac{d}{dx}(u(x)^p) = p(u(x))^{p-1} \times u'(x) \tag{11}$$

Derivative of $\log_b x$

If $b > 1$,

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b} \quad \forall x \in \mathbf{R}^+ \land x \neq 0 \tag{12}$$

$$\frac{d}{dx}(\log_b |x|) = \frac{1}{x \ln b} \quad \forall x \in \mathbf{R}^+ \land x \neq 0 \tag{13}$$

7.5 Inverse Trigonometric Functions

Derivative of Inverse Trigonometric Functions

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\quad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$, $\{x \in \mathbf{R} \mid -1 < x < 1\}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\quad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$, $\{x \in \mathbf{R} \mid -\infty < x < \infty\}$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$ $\quad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$, $\{x \in \mathbf{R} \mid |x| > 1\}$

Integrals Involving Inverse Trigonometric Functions

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C \tag{14}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C \tag{15}$$

$$\int \frac{dx}{x\sqrt{a^2 + x^2}} = \frac{1}{a} \sec^{-1} \left|\frac{x}{a}\right| + C \tag{16}$$

7.6 L'Hôpital's Rule and Growth Rates of Functions

Indeterminate forms $1^\infty, 0^0, \infty^0$

Assume $\lim_{num_1} f(x)^{g(x)}$ has the indeterminate form 1^∞, 0^0, or ∞^0.

1. Evaluate $L = \lim_{num_1} g(x) \ln f(x)$. This limit can be put in the form $0/0$ or ∞/∞, both of which are handled by l'Hôpital's rule.

2. Then $\lim_{num_1} f(x)^{g(x)} = e^L$

Growth Rates of Functions (as $x \to \infty$**)**

Suppose f and g are functions with $\lim_{num_1} f(x) = \lim_{num_1} g(x) = \infty$. Then f **grows faster than g** as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \quad \text{or, quantitatively, if} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \tag{17}$$

The functions f and g have **comparable growth rates** if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M$$

where $M \in \mathbb{R}^+$.

Ranking Growth Rates as $x \to \infty$

Let $f << g$ mean that g grows faster than f as $f \to \infty$. With positive real numbers p, q, r, s and $b > 1$,

$$\ln^q x << x^p << x^p \ln^r x << x^{p+s} << b^x << x^x \tag{18}$$

7.7 Hyperbolic Functions

Hyperbolic Functions

Hyperbolic Cosine

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{19}$$

Hyperbolic Sine

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{20}$$

Hyperbolic Tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{21}$$

Hyperbolic Cotangent

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \tag{22}$$

Hyperbolic Secant

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \tag{23}$$

Hyperbolic Cosecant

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \tag{24}$$

Hyperbolic Identities

$\cosh^2 x - \sinh^2 x = 1$	$\cosh(-x) = \cosh x$
$1 - \tanh^2 x = \operatorname{sech}^2 x$	$\sinh(-x) = -\sinh x$
$\coth^2 x - 1 = \operatorname{csch}^2 x$	$\tanh(-x) = -\tanh x$
$\coth(x + y) = \cosh x \cosh y + \sinh x \sinh y$	
$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$	
$\cosh 2x = \cosh^x + \sinh^2$	$\sinh 2x = 2 \sinh x \cosh x$
$\cosh^2 x = \frac{\cosh 2x+1}{2}$	$\sinh^2 x = \frac{\cosh 2x-1}{2}$

Derivatives and Integral Formulas

1. $\frac{d}{dx}(\cosh x) = \sinh x \quad \Rightarrow \quad \int \sinh x \; dx = \cosh x + C$
2. $\frac{d}{dx}(\sinh x) = \cosh x \quad \Rightarrow \quad \int \cosh x \; dx = \sinh x + C$
3. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \quad \Rightarrow \quad \int \operatorname{sech}^2 x \; dx = \tanh x + C$
4. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x \quad \Rightarrow \quad \int \operatorname{csch}^2 x \; dx = -\coth x + C$
5. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \quad \Rightarrow \quad \int \operatorname{sech} x \tanh x \; dx = -\operatorname{sech} x + C$
6. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x \quad \Rightarrow \quad \int \operatorname{csch} x \coth x \; dx = -\operatorname{csch} x + C$

Integrals of Hyperbolic Functions

1. $\int \tanh x \, dx = \ln \cosh x + C$
2. $\int \coth x \, dx = \ln|\sinh x| + C$
3. $\int \operatorname{sech} x \, dx = \tan^{-1}|\sinh x| + C$
4. $\int \operatorname{csch} x \, dx = \ln|\tanh(x/2)| + C$

Inverses of the Hyperbolic Functions Expressed as Logarithms

$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \; (x \geq 1)$ $\quad \operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x} \; (0 < x \leq 1)$

$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$ $\quad \operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x} \; (x \neq 0)$

$\tanh^{-1} x = \frac{1}{2} \ln(\frac{1+x}{1-x}) \; (|x| < 1)$ $\quad \coth^{-1} x = \tanh^{-1} \frac{1}{x} \; (|x| > 1)$

Derivatives of the Inverse Hyperbolic Functions

$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}} \; (x > 1)$ $\quad \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$

$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \; (|x| < 1)$ $\quad \frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2} \; (|x| > 1)$

$\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}} \; (0 < x < 1)$ $\quad \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{1+x^2}} \; (x \neq 0)$

Integral Formulas

1. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1} \frac{x}{a} + C,$ for $x > a$
2. $\int \frac{dx}{x^2+a^2} = \sinh^{-1} \frac{x}{a} + C,$ for all x

3. $\int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C,\ \text{for} |x| < a = \frac{1}{a}\coth^{-1}\frac{x}{a} + C, \text{for } |x| > a$

4. $\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\,\text{sech}^{-1}\frac{x}{a} + C,\ \text{for } 0 < x < a$

5. $\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\,\text{csch}^{-1}\frac{|x|}{a} + C,\ \text{for } x \neq 0$

Chapter 8: Integration Techniques

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8.1 Basic Approaches

$$\int k\,dx = kx + C \tag{1}$$

$$\int k^p\,dx = \frac{k^{p+1}}{p+1} + C, p \in \mathbf{R} \wedge \neq -1 \tag{2}$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C \tag{3}$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C \tag{4}$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C \tag{5}$$

$$\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C \tag{6}$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C \tag{7}$$

$$\int \csc ax \cot ax\,dx = -\frac{1}{a}\csc ax \cot ax + C \tag{8}$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C \tag{9}$$

$$\int \frac{dx}{x} = \ln|x| + C \tag{10}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C \tag{11}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \tag{12}$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C \tag{13}$$

8.2 Integration By Parts

Suppose that u and v are differentiable functions. Then

$$\int u\,dv = uv - \int v\,du \tag{14}$$

Integration by Parts for Definite Integrals

Let u and v be differentiable. Then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx \tag{15}$$

Integral of $\ln x$

$$\int \ln x\,dx = x\ln x - x + C \tag{16}$$

8.3 Trigonometric Integrals

$\int \sin^m x \cos^n x \; dx$ **Strategy**

m **is odd,** n **real** Split off $\sin x$, rewrite the resulting even power of $\sin x$ in terms of $\cos x$, and then use $u = \cos x$

n **odd,** m **real** Split off $\cos x$, rewrite the resulting even power of $\cos x$ in terms of $\sin x$, and then use $u = \sin x$.

m **and** n **both even, nonnegative** Use half-angle identities to transform the integrand into a polynomial in $\cos 2x$, and apply the preceding strategies once again to powers of $\cos 2x$ greater than 1.

Reduction Formulas

Assume n is a positive integer.

$$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \; dx \tag{17}$$

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \; dx \tag{18}$$

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \; dx, n \neq 1 \tag{19}$$

$$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \; dx, n \neq 1 \tag{20}$$

Integrals of $\tan x, \cot x, \sec x,$ and $\csc x$

$$\int \tan x \, dx = -\ln|\cos x| + C = \ln|\sec x| + C \tag{21}$$

$$\int \cot x \, dx = \ln|\sin x| + C \tag{22}$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C \tag{23}$$

$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C \tag{24}$$

$\int \tan^m x \sec^n x \, dx$ **Strategy**

n **even** Split off $\sec^2 x$, rewrite the remaining even power of $\sec x$ in terms of $\tan x$, and use $u = \tan x$.

m **odd** Split off $\sec x \tan x$, rewrite the remaining even power of $\tan x$ in terms of $\sec x$, and use $u = \sec x$.

m **even and** n **odd** Rewrite the even power of $\tan x$ in terms of $\sec x$ to produce a polynomial in $\sec x$; apply reduction formula 4 to each term.

8.4 Trigonometric Substitutions

The Integral Contains...	Corresponding Substitution	Useful Identity
$a^2 - x^2$	$x = a\sin\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \forall \lvert x \rvert \leq a$	$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$a^2 + x^2$	$x = a\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$x^2 - a^2$	$x = a\sec\theta,$ $\begin{cases} 0 \leq \theta < \frac{\pi}{2}, \forall x \geq a \\ \frac{\pi}{2} < \theta \leq \pi, \forall x \leq -a \end{cases}$	$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

8.5 Partial Fractions

Partial Fractions with Simple Linear Factors

Suppose $f(x) = p(x) > q(x)$, where p and q are polynomials with no common factors and with the degree of p less than the degree of q. Assume that q is the product of simple linear factors. The partial fraction decomposition is obtained as follows.

Step 1. Factor the denominator q in the form $(x-r_1)(x-r_2)\cdots(x-r_n)$, where $r_1, \ldots, r_n$ are real numbers.

Step 2. Partial fraction decomposition Form the partial fraction decomposition by writing

$$\frac{p(x)}{q(x)} = \frac{A_1}{x-r_1} + \frac{A_2}{x-r_2} + \cdots + \frac{A_n}{x-r_n} \tag{25}$$

Step 3. Clear denominators Multiply both sides of the equation in Step 2 by $q(x) = (x-r_1)(x-r_2)\cdots(x-r_n)$, which produces conditions for $A_1, \ldots, A_n$.

Step 4. Solve for coefficients Equate like powers of x in Step 3 to solve for the undertmined coefficients $A_1, \ldots, A_n$

Partial Fractions For Repeated Linear Factors

Suppose the repeated linear factor $(x-r)^m$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition has a partial fraction for each power of $(x-r)$ up to and including the mth power; that is, the partial fraction decomposition contains the sum

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m} \tag{26}$$

where $A_1, \ldots, A_m$ are constants to be determined.

Partial Fractions with Simple Irreducible Quadratic Factors

Suppose a simple irreducible factor ax^2+bx+c appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition contains a term of the form

$$\frac{Ax+B}{ax^2+bx+c} \tag{27}$$

where A and B are unknown coefficients to be determined.

Partial Fraction Decomposition

Let $f(x) = p(x)/q(x)$ be a proper rational function in reduced form. Assume the denominator q has been factored completely over the real numbers and m is a positive integer.

Simple linear factor A factor $x - r$ in the denominator requires the partial fraction $\frac{A}{x-r}$.

Repeated linear factor A factor $(x - r)^m$ with $m > 1$ in the denominator requires the partial fractions.

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m} \tag{28}$$

Simple irreducible quadratic factor An irreducible factor $ax^2 + bx + c$ in the denominator requires the partial fraction

$$\frac{Ax+B}{ax^2+bx+c} \tag{29}$$

Repeated irreducible quadratic factor An irreducible factor $(ax^2 + bx + c)^m$ with $m > 1$ in the denominator requires the partial fractions

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m} \tag{30}$$

8.8 Improper Integrals

Improper Integrals over Infinite Intervals

1. If f is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx \tag{31}$$

provided the limit exists.

2. If f is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx \tag{32}$$

provided the limit exists.

3. If f is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx \tag{33}$$

provided both limits exist, where c is any real number.

In each case, if the limit exists, the improper integral is said to **converge**, if it does not exist, the improper integral is said to **diverge**.

Improper Integrals with an Unbounded Integrand

1. Suppose f is continuous on $(a, b]$ with $\lim_{num_1} f(x) = \pm\,\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx \tag{34}$$

provided the limit exists.

2. Suppose f is continuous on $[a, b)$ with $\lim_{num_1} f(x) = \pm\,\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx \tag{35}$$

provided the limit exists.

3. Suppose f is continuous on $[a, b]$ except at the interior point p where f is unbounded. Then

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx, \tag{36}$$

provided the improper integrals on the right side exist.

In each case, if the limit exists, the improper integrals is said to **converge**, if it does not exists, the improper integral is said to **diverge**.

Chapter 9: Sequences and Infinite Series

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9.1 An Overview

Sequence

A **sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\} \tag{1}$$

A sequence may be generated by a **recurrence relations** of the form $a_{n+1} = f(a_n)$, for $n = 1, 2, 3, \ldots$, where a_1 is given. A sequence may also be defined with an **explicit form** of the form $a_n = f(n)$, for $n = 1, 2, 3, \ldots$.

Limit of a Sequence

If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases, then we say $\lim_{num_1} = L$ exists, and the sequence **converges** to L. If the terms of the sequence do not approach a single number as n increases, the sequence has no limits, and the sequence **diverges**.

Infinite Series

Given a set of numbers $\{a_1, a_2, a_3, \ldots\}$, the sum

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k \tag{2}$$

is called an **infinity series**. Its **sequence of partial sums** $\{S_n\}$ has the terms

$$S_1 = a_1 \tag{3}$$

$$S_2 = a_1 + a_2 \tag{4}$$

$$S_3 = a_2 + a_2 + a_3 \tag{5}$$

$$\vdots \tag{6}$$

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \quad n = 1, 2, 3, \ldots \tag{7}$$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series **converges** to that limits, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L \tag{8}$$

If the sequence of partial sums diverges, the infinite series also **diverges**.

9.2 Sequences

Limits of Sequences from Limits of Functions

Suppose f is a function such that $f(n) = a_n$ for all positive integers n. If $\lim\limits_{num_1} = L$, then the limits of the sequences $\{a_n\}$ is also L.

Properties of Limits of Sequences

Assume that the sequence $\{a_n\}$ and $\{b_n\}$ have limits A and B, respectively. Then,

1. $\lim\limits_{num_1} (a_n \pm b_n) = A \pm B$
2. $\lim\limits_{num_1} ca_n = cA$, where c is a real number
3. $\lim\limits_{num_1} a_n b_n = AB$
4. $\lim\limits_{num_1} \frac{a_n}{b_n} = \frac{A}{B}$, provided $B \neq 0$.

Geometric Sequences

Let r be a real number. Then,

$$\lim_{n \to \infty} r^n = \begin{cases} 0 & \Leftrightarrow |r| < 1 \\ 1 & \Leftrightarrow r = 1 \\ \text{does not exist} & \Leftrightarrow r \leq -1 \vee r > 1 \end{cases} \tag{9}$$

Squeeze Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences with $a_n \leq b_n \leq c_n$ for all n greater than some index N. If $\lim\limits_{num_1} a_n = \lim\limits_{num_1} c_n = L$, the $\lim\limits_{num_1} b_n = L$.

Bounded Monotonic Sequences

A bounded monotonic sequence converges.

Growth Rates of Sequences

The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$

$$\{\ln^q n\} \ll \{n^p\} \ll \{n^p \ln^r n\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\} \tag{10}$$

The ordering applies for $p, q, r, s, b \in \mathbb{R}^+ \wedge b > 1$.

Limit of a Sequence

The sequence $\{a_n\}$ converges to L provided the terms of a_n can be made arbitrarily close to L by taking n sufficiently large. More precisely, $\{a_n\}$ has the unique limit L if given any tolerance $\epsilon > 0$, it is possible to find a positive integer N (depending only on ϵ) such that

$$|a_n - L| < \epsilon \quad \text{whenever } n > N \tag{11}$$

if the **limit of a sequence** is L, we say the sequence **converges** to L, written

$$\lim_{n\to\infty} a_n = L \tag{12}$$

A sequence that does not converge is said to **diverge.**

9.3 Infinite Series

Geometric Series

$$\sum_{k=0}^{n-1} ar^k = S_n = a\frac{1-r^n}{1-r} \tag{13}$$

Geometric Series

Let $a \neq 0$ and r be real numbers. If $|r| < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. If $|r| \geq 1$, then the series diverges.

9.4 The Divergence and Integral Tests

Divergence Test

If $\sum a_k$ converges, then $\lim\limits_{n u m_1} a_k = 0$. Equivalently, if $\lim\limits_{n u m_1} a_k \neq 0$, then the series diverges. However, this cannot be used to prove convergence. If $\lim\limits_{n u m_1} a_k = 0$, the test is inconclusive.

Harmonic Series

The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ diverges, even though the terms of the series approach zero.

Integral Test

Suppose f is a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Then

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \int_1^{\infty} f(x)\,dx \tag{14}$$

either both converge or both diverge. In the case of convergence, the value of the integral is *not*, in general, equal to the value of the series.

Convergence of the p-Series

The p-Series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges, for $p > 1$, and diverges for $p \leq 1$.

Estimating Series with Positive Terms

Let f be continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Let $S = \sum_{k=1}^{\infty} a_k$ be a convergence series and let $S_n = \sum_{k=1}^{n}$ be the sum of of the first n terms of the series. The remainder $R_n = S - S_n$ satisfies

$$R_n \leq \int_n^{\infty} f(x)\,dx \tag{15}$$

Furthermore, the exact value of the series is bounded as follows:

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \leq \sum_{k=1}^{\infty} a_k \leq S_n + \int_n^{\infty} f(x)\,dx. \tag{16}$$

Properties of Convergent Series

1. Suppose $\sum a_k$ converges to A and let c be a real number. The series $\sum ca_k$ converges and $\sum ca_k = c\sum a_k = cA$

2. Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B. The series $\sum(a_k \pm b_k)$ converges and $\sum(a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$

3. *Whether* a series converges does not depend on a fininte number of terms added to or removed form the series. Specifically, if M is a positive integer, then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=M}^{\infty} a_k$ both converge or both diverge. However, the *value* of a convergent series does change if nonzero terms are added or deleted.

9.5 The Ratio, Root, and Comparison Tests

Useful Identities

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^k = e \tag{17}$$

$$\lim_{k\to\infty} k^{\frac{1}{k}} = 1 \tag{18}$$

The Ratio Test

Let $\sum a_k$ be an infinite series with positive terms and let $r = \lim_{num_1} \frac{a_{k+1}}{a_k}$.

1. If $0 \leq r < 1$, the series converges.
2. If $r > 1$ (including $r = \infty$), the series diverges.
3. If $r = 1$, the test is inconclusive.

The Root Test

Let $\sum a_k$ be an infinite series with nonegative terms and let $\rho = \lim_{num_1} \sqrt[k]{a_k}$.

1. If $0 \leq \rho < 1$, the series converges.
2. If $\rho > 1$ (including $\rho = \infty$), the series diverges.
3. If $\rho = 1$, test is inconclusive.

The Comparison Test

Let $\sum a_k$ and $\sum b_k$ be a series with postivie terms.

1. If $0 < a_k \leq b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges.
2. If $0 < b_k \leq a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

The Limit Comparison Test

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L \tag{19}$$

- If $0 < L < \infty$ (that is, L is a finite, positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.
- If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.
- If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Guidelines

- Begin with the Divergence Test. If you show that $\lim_{num_1} a_k \neq 0$, then the series diverges and your work is finished.
- Is the series a special series? Recall the convergence properties for the following series:
  - Geometric series: $\sum ar^k$ converges for $|r| < 1$ and diverges for $|r| \geq 1$ $(a \neq 0)$.
  - p-series: $\sum \frac{1}{k^p}$ converges for $p > 1$, and diverges for $p \leq 1$.
  - Check also for telescoping series.
- If the general kth term of the series looks like a function you can integrate, then try the Integral Test.
- If the general kth term of the series involves $k!, k^k,$ or a^k, where a is a constant, the Ratio Test is advisable. Series with k in an exponent may yield to the Root Test.
- If the general kth term of the series is a rational function of k (or a root of a rational function), use the Comparison or the Limit Comparison Test. Use the families of series given in Step 2 as comparison series.

9.6 Alternating Series

The Alternating Series Test

The alternating series $\sum(-1)^{k+1}a_k$ converges provided

1. the terms of the series are non-increasing in magnitude ($0 < a_{k+1} \leq a_k$, for k greater than some index N) and

2. $\lim\limits_{num_1} = 0$

Alternating Harmonic Series

The alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ converges (even though the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ diverges).

Remainder in Alternating Series

Let $R_n = |S - S_n|$ be the remainder in approximating the value of a convergent alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ by the sum of its first n terms. Then $R_n \leq a_{n+1}$. In other words, the remainder is less than or equal to the magnitude of the first neglected term.

Absolute and Conditional Convergence

Assume the infinite series $\sum a_k$ converges. The series $\sum a_k$ **converges absolutely** if the series $\sum |a_k|$ converges. Otherwise, the series $\sum a_k$ **converges conditionally**.

Absolute Convergence Implies Convergence

If $\sum |a_k|$ converges, then $\sum a_k$ converges (absolute convergence implies convergence). If $\sum a_k$ diverges, then $\sum |a_k|$ diverges.

Series or Test	Form of Series	Condition for Convergence	Condition for Divergence	Comments
Geometric Series	$\sum_{k=0}^{\infty} ar^k, a \neq 0$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	If $\lvert r\rvert < 1$, then $\sum a_{k=1}^{\infty} ar^k = \frac{a}{1-r}$
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does Not Apply	$\lim_{k\to\infty a_k \neq 0}$	Cannot be used to prove convergence.
Integral Test	$\sum a_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing.	$\int_1^{\infty} f(x)dx < \infty$	$\int_1^{\infty} f(x)dx$ does not exist	The value of the integral is not the value of the series.
p-Series	$\sum a_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests.
Ratio Test	$\sum a_{k=1}^{\infty}$ where $a_k > 0$	$\lim_k \to \infty \frac{a_{k+1}}{a_k} < 1$	$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} > 1$	Inconclusive if $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$
Root Test	$\sum a_{k=1}^{\infty}$ where $a_k \geq 0$	$\lim_{k\to\infty} \sqrt[k]{a_k} < 1$	$\lim_{k\to\infty} \sqrt[k]{a_k} > 1$	Inconclusive if $\lim_{k\to\infty} \sqrt[k]{a_k} = 1$
Comparison Test	$\sum a_{k=1}^{\infty}$ where $a_k > 0$	$0 < a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges	$0 < b_k \leq a_k$ and $\sum a_{k=1}^{\infty} b_k$ diverges	$\sum a_{k=1}^{\infty} a_k$ is given; you supply $\sum a_{k=1}^{\infty} b_k$
Limit Comparison Test	$\sum a_{k=1}^{\infty} a_k$ where $a_k > 0, b_k > 0$	$0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges	$\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum a_{k=1}^{\infty} b_k$ diverges.	$\sum a_{k=1}^{\infty} a_k$ is given; you supply $\sum a_{k=1}^{\infty} b_k$
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0, 0 < a_{k+1} \leq a_k$	$\lim_{k\to\infty} a_k = 0$	$\lim_{k\to\infty} a_k \neq 0$	Remainder R_n satisfies $R_n \leq a_{n+1}$
Absolute Convergence	$\sum a_{k=1}^{\infty} a_k, a_k$ arbitrary	$\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges.	Applies to arbitrary series	

Chapter 10: Power Series

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10.1 Approximating Functions with Polynomials

Taylor Polynomials

Let f be a function with $f', f'', \ldots, f^{(n)}$ defined at a. The n**th-order Taylor polynomial** for f with its **center** at a, denoted p_n, has the property that it matches f in value, slope, and all derivatives up to the nth derivative at a; that is,

$$p_n(a) = f(a), p_n'(a) = f'(a), \ldots, p_n^{(n)}(a) = f^{(n)}(a). \tag{1}$$

The nth-order Taylor polynomial centered at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{2}$$

More compactly, $p_n(x) = \sum_{k=0}^{n} c_k (x-a)^k$, where the **coefficients** are

$$c_k = \frac{f^{(k)}(a)}{k!}, \quad \text{for } k = 0, 1, 2, \ldots, n \tag{3}$$

Remainder in a Taylor Polynomial

Let p_n be the Taylor polynomial of order n for f. The **remainder** in using p_n to approximate f at the point x is

$$R_n(x) = f(x) - p_n(x) \tag{4}$$

Taylor's Theorem

Let f have continuous derivatives up to $f^{(n+1)}$ on an open interval I containing a. For all x in I,

$$f(x) = p_n(x) + R_n(x), \tag{5}$$

where p_n is the nth-order Taylor polynomial for f centered at a, and the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{(n+1)}, \tag{6}$$

for some point c between x and a.

Estimate of the Remainder

Let n be a fixed positive integer. Suppose there exists a number M such that $|f^{(n+1)}(c)| \leq M$, for all c between a and x inclusive. The remainder in the nth-order Taylor polynomial for f centered at a satisfies

$$|R_n(x)| = |f(x) - p_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!} \tag{7}$$

10.2 Properties of Power Series

Power Series

A **power series** has the general form

$$\sum_{k=0}^{\infty} c_k(x-a)^k \tag{8}$$

where a and c_k are real numbers, and x is a variable. The c_k's are the **coefficients** of the power series and a is the **center** of the power series. The set of values of x for which the series converges is its **interval of convergence**. The **radius of convergence** of the power series, denoted R, is the distance from the center of the series to the boundary of the interval of convergence.

Convergence of Power Series

A power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ centered at a converges in one of three ways:

1. The series converges absolutely for all x, in which case the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.
2. There is a real number $R > 0$ such that the series converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$, in which case the radius of convergence is $R = 0$.
3. The series converges only at a, in which case the radius of convergence is $R = 0$.

Combining Power Series

Suppose the power series $\sum c_k x^k$ and $\sum d_k x^k$ converge absolutely to $f(x)$ and $g(x)$, respectively, on an interval I.

1. **Sum and difference:** The power series $\sum (c_k \pm d_k)x^k$ converges absolutely to $f(x) \pm g(x)$ on I.
2. **Multiplication by a power:** The power series $x^m \sum c_k x^k = \sum c_k x^{k+m}$ converges absolutely to $x^m f(x)$ on I, provided m is an integer such that $k + m \geq 0$ for all terms of the series.

3. **Composition:** If $h(x) = bx^m$, where m is a positive integer and b is a real number, the power series $\sum c_k(h(x))^k$ converges absolutely to the composite function $f(h(x))$, for all x such that $h(x)$ is in I.

Differentiating and Integrating Power Series

Let the function f be defined by the power series $\sum c_k(x-a)^k$ on its interval of convergence I.

- f is a continuous function on I.
- The power series may be differentiated or integrated term by term, and the resulting power series converges to $f'(x)$ or $\int f(x)\, dx + C$, respectively, at all points in the interior of I, where C is an arbitrary constant.

10.3 Taylor Series

Taylor/Maclaurin Series for a Function

Suppose the function f has derivatives of all orders on an interval containing the point a. The **Taylor series for f centered at a** is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots \tag{9}$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{10}$$

A Taylor series centered at 0 is called a **Maclaurin series**.

Binomial Coefficients

$\forall p, k \in \mathbb{R} \wedge k \geq 1$

$$\binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}. \tag{11}$$

With the special case of $\binom{p}{0} = 1$.

Binomial Series

$\forall p \in \mathbb{R} \wedge p \neq 0$, the Taylor series for $f(x) = (1+x)^p$ centered at 0 is the **binomial series**

$$\sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!} x^k \tag{12}$$

$$= 1 + px + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \tag{13}$$

The series converges for $|x| < 1$ (and possibly at the endpoints, depending on p). If p is a nonnegative integer, the series terminates and results in a polynomial of degree p.

Convergence of Taylor Series

Let f have derivatives of all orders on an open interval I containing a. The Taylor series for f centered at a converges to f , for all x in I, if and only if $\lim_{n\to\infty} R_n(x) = 0$, for all x in I, where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \tag{14}$$

is the remainder at x (with c between x and a).

Taylor Series Functions

Chapter 11: Parametric and Polar Curves

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11.1 Parametric Equations

Forward or Positive Orientation

The direction in which a parametric curve is generated as the parameter increases is called the **forward**, or **positive**, **orientation** of the curve.

Derivative for Parametric Curves

Let $x = g(t)$ and $y = h(t)$, where g and h are differentiable on an interval $[a, b]$. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} \tag{1}$$

provided $\frac{dx}{dt} \neq 0$.

11.2 Polar Coordinates

Introduction

- Up to now we have only studied in a Cartesian coordinate system.
  - A Cartesian coordinate system is just a plane described by Cartesian (or, algebraic) equations and points in a finite dimensions.
    * *One Dimension*: Lines.
    * *Two Dimensions*: x^2.
    * *Three, Four*: Upper-level Calculus and Physics.
- Let's define an alternative coordinate system — **polar coordinate.**
  - coordinates are constants on circles and rays.
  - Useful for navigation, position, and gravitation fields.

Defining Polar Coordinates

Pole The origin of the coordinate system.

Polar Axis Synonymous for the positive x-axis.

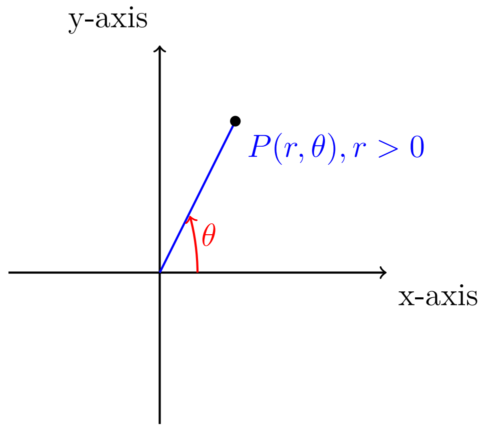
Polar Coordinates A polar coordinates P has the form (r, θ).

> **Radial Coordinate** The radial coordinate r describes the *signed*, or *directed*, distance from the origin to P.
>
> **Angular Coordinate** The angular coordinate θ describes an angle whose initial side is the positive x-axis and whose terminal side lies on the ray passing through the origin and P.

Notes

- **Positive angle measurements are measured *counterclockwise* from the origin.**
- Every point has multiple representations.
  - Angles are periodic, so multiples of 2π gives the same angle.

– Coordinates may be negative. So (r, θ) can be represented as $(-r, \theta + \pi)$ and $(-r, \theta - \pi)$



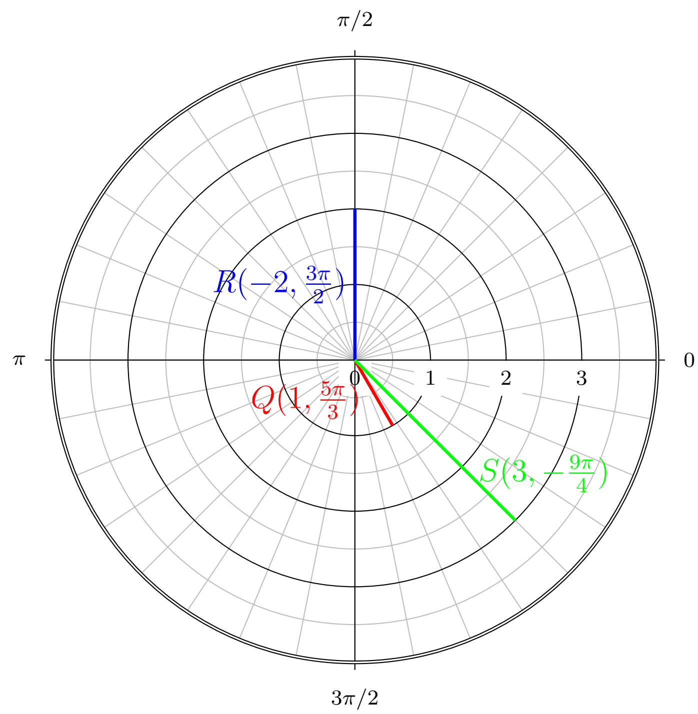
In summary,

- $(r, \theta + 2\pi)$ represents the same point as (r, θ)
- $P(r, \theta)$ and $P'(-r, \theta)$ are reflections through the origin.

Examples

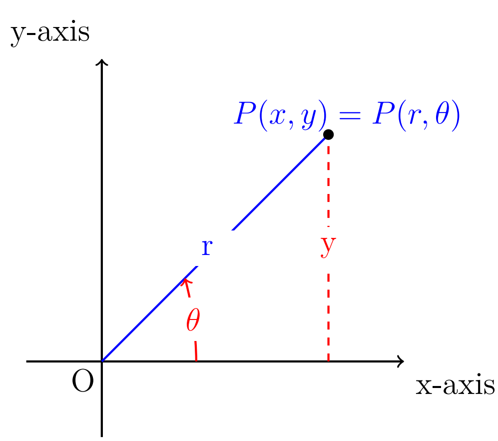
Graph the following points in polar coordinates:

- $Q(1, \frac{5\pi}{3})$
- $R(-2, \frac{3\pi}{2})$
- $S(3, -\frac{9\pi}{4})$
  - Now give two alternative representations.
  - $S'(3, \frac{1\pi}{4})$
  - $S''(-3, -\frac{5\pi}{4})$



Converting Between Cartesian and Polar Coordinates

- We sometimes need to convert between Cartesian and polar coordinates.
- Let's turn this problem into a right triangle.



A point with polar coordinates (r, θ) has Cartesian coordinates (x, y), where

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \tag{2}$$

A point with Cartesian coordinates (x, y) has polar coordinates (r, θ), where

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \tag{3}$$

Examples

BE SURE TO GRAPH POINTS IN CARTESIAN FIRST.

Polar to Cartesian

Express the point with the following polar coordinates in Cartesian coordinates: $P(3, \frac{2\pi}{3})$

$$\begin{aligned} x &= r\cos\theta \\ &= 3\cos({}^{2\pi}/_{3}) \\ &= -3({}^{1}/_{2}) \\ &= -{}^{3}/_{2} \end{aligned} \qquad \begin{aligned} y &= r\sin\theta \\ &= 3\sin({}^{2\pi}/_{3}) \\ &= 3({}^{\sqrt{3}}/_{2}) \\ &= {}^{3\sqrt{3}}/_{2} \end{aligned}$$

Polar to Cartesian

Express the point with the following polar coordinates in Cartesian coordinates: $Q(e, -\frac{\pi}{4})$

$$\begin{aligned} x &= r\cos\theta \\ &= e\cos(-{}^{\pi}/_{4}) \\ &= e({}^{\sqrt{2}}/_{2}) \\ &= {}^{e\sqrt{2}}/_{2} \end{aligned} \qquad \begin{aligned} y &= r\sin\theta \\ &= e\sin(-{}^{\pi}/_{4}) \\ &= -e({}^{\sqrt{2}}/_{2}) \\ &= -{}^{e\sqrt{2}}/_{2} \end{aligned}$$

Cartesian To Polar

Express the point with the following Polar coordinates to Cartesian coordinates: $R(1, -1)$

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + (-1)^2} \\ &= \sqrt{2} \end{aligned} \qquad \begin{aligned} \tan\theta &= {}^{y}/_{x} \\ &= {}^{-1}/_{1} \\ &= -1 \\ \theta &= {}^{-\pi}/_{4} \text{ or } {}^{7\pi}/_{4}. \end{aligned}$$

Therefore, two possible solutions are: $(\sqrt{2}, {}^{-\pi}/_{4})$ or $(\sqrt{2}, {}^{7\pi}/_{4})$

Cartesian To Polar

Express the point with the following Polar coordinates to Cartesian coordinates: $S(1, \sqrt{3})$

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(\sqrt{3})^2 + (1)^2} \\ &= 2 \end{aligned} \qquad \begin{aligned} \tan\theta &= {}^{y}/_{x} \\ &= {}^{\sqrt{(3)}}/_{1} \\ \theta &= {}^{\pi}/_{3} \text{ or } {}^{4\pi}/_{3}. \end{aligned}$$

Therefore, two possible solutions are: $(2, {}^{\pi}/_{3})$ or $(2, {}^{4\pi}/_{3})$.

Basic Curves in Polar Coordinates

- A curve in polar coordinates is the set of **points** that satisfy an equation in r and θ.
- This makes graphing some things easier than others.
- Look at $r = 3$ is the set of all points that satisfy being away from the origin of 3 units.
  - This is because θ is not specified, it's arbitrary. Basically, θ is the function.

– In general, $r = a$, $\forall a \in \mathbb{R}^+$ describes a circle.

- Taking the converse, let r be arbitrary.
  - If the r is arbitrary, and we specify the angle, what do you think we get?
  - A line!
  - Take $\sqrt{2}/2$.

Polar to Cartesian Graph Example

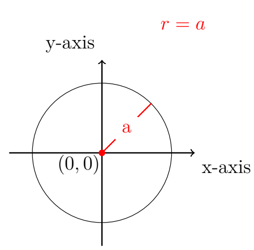
Convert the polar equation $r = 6\sin\theta$ to Cartesian coordinates and describe the corresponding graph.

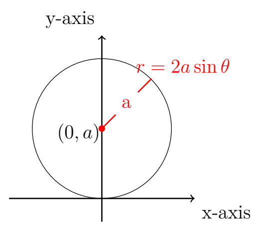
$$\begin{aligned} r^2 &= 6r\sin\theta && (4)\\ x^2 + y^2 &= 6y && (5)\\ 0 &= x^2 + y^2 - 6y && (6)\\ &= x^2 + (y^2 - 6y + 9) - 9 && (7)\\ &= x^2 + (y-3)^2 - 9 && (8) \end{aligned}$$

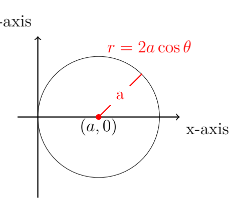
We recognize this to be the equation of a circle, centered at $(0, 3)$ at 3. We can also generalize this.

Circle in Polar Coordinates

- The equation $r = a$ describes a circle of radius $|a|$ centered at $(0, 0)$.
- The equation $r = 2a\sin\theta$ describes a circle of radius $|a|$ centered at $(0, a)$.
- The equation $r = 2a\cos\theta$ describes a circle of radius $|a|$ centered at $(a, 0)$.



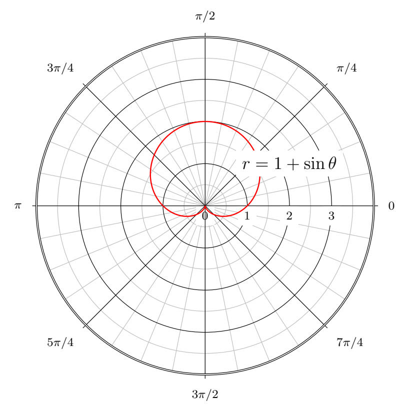




Graphing In Polar Coordinates

Graph the polar equation $r = f(\theta) = 1 + \sin\theta$

θ	$r = 1 + sin\theta$
0	1
$^{\pi}/_{6}$	$^{3}/_{2}$
$^{\pi}/_{2}$	2
$^{5\pi}/_{6}$	$^{3}/_{2}$
π	1
$^{7\pi}/_{6}$	$^{1}/_{2}$
$^{3\pi}/_{2}$	0
$^{11\pi}/_{6}$	$^{1}/_{2}$
2π	1



The resulting curve is known as a **cardioid**.

Cartesian-to-Polar Method for Graphing $r = f(\theta)$

1. Graph $r = f(\theta)$ *as if* r *and* θ *were Cartesian coordinates* with θ on the horizontal axis and r on the vertical axis. Be sure to choose an interval in θ on which the entire polar curve is produced.

2. Use the Cartesian graph in Step 1 as a guide to sketch the points (r, θ) on the final *polar* curve.

Example

With the alternate graphing method, graph $r = 1 + \sin\theta$.

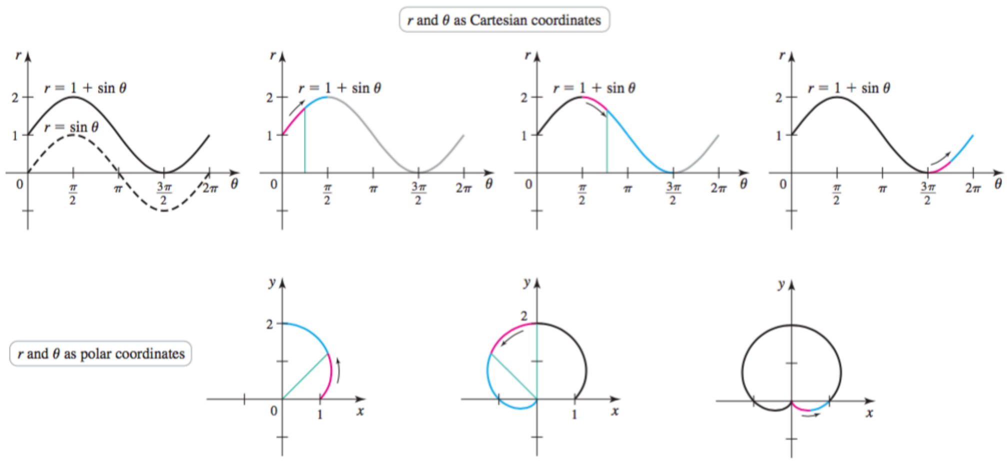


FIGURE 11.26

Symmetry In Polar Equations

Symmetry about the x-axis occurs if the point (r, θ) is on the graph whenever $(r, -\theta)$ is on the graph.

Symmetry about the y-axis occurs if the point (r, θ) is on the graph whenever $(r, \pi - \theta) = (-r, -\theta)$ is on the graph.

Symmetry about the origin occurs if the point $(r, \theta$ is on the graph whenever $(-r, \theta) = (r, \theta + \pi)$ is on the graph.

11.3 Calculus in Polar Coordinates

Slope of a Tangent Line

Let f be a differentiable function at θ_0. The slope of the line tangent to the curve $r = f(\theta)$ at the point $(f(\theta_0, \theta_0)$ is

$$\frac{dy}{dx} = \frac{f'(\theta_0)\sin\theta_0 + f(\theta_0)\cos\theta_0}{f'(\theta_0)\cos\theta_0 - f(\theta_0)} \tag{9}$$

Area of Regions in Polar Coordinates

Let R be the region bounded by the graphs of $r = f(\theta)$ and $r = g(\theta)$, between $\theta = \alpha$ and $\theta = \beta$, where f and g are continous and $f(\theta) \geq g(\theta) \geq 0$ on $[\alpha, \beta]$. The area of R is

$$\int_\alpha^\beta \frac{1}{2}(f(\theta)^2 - g(\theta)^2)\, d\theta \tag{10}$$

Calculus II: Complete Notes

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Contents

6 Applications of Integration

6.1 Physical Applications

Mass of a One-Dimensional Object

Suppose a thin bar or wire is represented by a line segment on the interval $a \leq x \leq b$ with a density function ρ (with units of mass per length). The mass of the object is

$$m = \int_a^b \rho(x)\,dx \tag{1}$$

Work

The work done by a variable force F in moving an object along a line from $x = a$ to $x = b$ in the direction of the force is

$$W = \int_a^b F(x)\,dx \tag{2}$$

Solving Lifting Problems

1. Draw a y-axis in the vertical direction (parallel to gravity) and choose a convenient origin. Assume the interval $[a, b]$ corresponds to the vertical extent of the fluid.

2. For $a \leq y \leq b$, find the cross-sectional area $A(y)$ of the horizontal slices and the distance $D(y)$ the slices must be lifted.

3. The work required to lift the water is

$$W = \int_a^b \rho g A(y) D(y)\,dy \tag{3}$$

Solving Force/Pressure Problems

1. Draw a y-axis on the face of the dam in the vertical direction and choose a convenient origin (often taken to be the base of the dam).

2. Find the width function $w(y)$ for each value of y on the face of the dam.

3. If the base of the dam is at $y = 0$ and the top of the dam is at $y = a$, then the total force on the dam is

$$F = \int \rho g(a - y)w(y)\, dy \tag{4}$$

7 Logarithmic, Exponential, and Inverse Trigonometric Functions

7.1 Inverse Function

Derivative of the Inverse Function

Let f be differentiable and have an inverse on an interval I. If x_0 is a point of I at which $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}, \text{ where } y_0 = f(x_0) \tag{5}$$

7.2 The Natural Logarithmic and Exponential Functions

The Natural Logarithm

The **natural logarithm** of a number $x > 0$, denoted x, is defined

$$\ln x = \int_1^x \frac{dt}{t} \tag{6}$$

Properties of the Natural Logarithm

1. The domain and range of $\ln x$ are $(0, \infty)$ and $(-\infty, \infty)$, respectively.
2. $\ln(xy) = \ln x + \ln y, \forall x, y \in \mathbf{R}^+$
3. $\ln(x/y) = \ln x - \ln y, \forall x, y \in \mathbf{R}^+$
4. $\ln x^p = p \ln x, \forall x \in \mathbf{Q}^+$
5. $\frac{d}{dx}(\ln |x|) = \frac{1}{x}, \forall x \in \mathbf{R} \wedge x \neq 0$
6. $\frac{d}{dx}(\ln |u(x)|) = \frac{u'(x)}{u(x)}$
7. $\int \frac{dx}{x} = \ln |x| + C$

The Number e

The number e is the real number that satisfies

$$\ln e = \int_1^e \frac{dt}{t} = 1 \tag{7}$$

The Exponential Function

$\forall x, y \in \mathbf{R}$

$$y = e^x \Leftrightarrow x = \ln y \tag{8}$$

Properties of e

- $e^{x+y} = e^x e^y$
- $e^{x-y} = e^x/e^y$
- $(e^x)^y = e^{xy}, \forall y \in \mathbf{Q}$
- $\ln(e^x) = x, \forall x \in \mathbf{R}$
- $e^{\ln\ x} = x, \forall x \in \mathbf{R}^+$

Exponential Functions with General Bases

Let $b \in \mathbf{R}^+ \wedge b \neq 1$. $\forall x \in \mathbf{R}^+$,

$$b^x = e^{x \ln b} \tag{9}$$

Derivative and Integral of the Exponential Function

$\forall x \in \mathbf{R}$,

$$\frac{d}{dx}(e^{u(x)}) = e^{u(x)}u'(x) \tag{10}$$

$$\int e^x \, dx = e^x + C \tag{11}$$

7.3 Logarithmic and Exponential Functions with Other Bases

Logarithmic Function Base b

For any base $b > 0$, with $b \neq 1$, the **logarithmic function base** b, denoted $\log_b x$, is the inverse of the exponential function b^x.

Inverse Relations for Exponential and Logarithmic Functions

For any base $b > 0$, with $b \neq 1$, the following inverse relation holds.

- $b^{\log_b x} = x, \forall x \in \mathbf{R}^+$
- $\log_b b^x = x, \forall x$

Derivative of b^x

If $b > 0 \land b \neq 1, \forall x$,

$$\frac{d}{dx}(b^x) = b^x \ln b \tag{12}$$

Indefinite Integral of b^x

For $b > 0 \land b \neq 1$,

$$\int b^x dx = \frac{1}{\ln b} b^x + C \tag{13}$$

General Power Rule

$\forall p, x \in \mathbf{R}^+$,

$$\frac{d}{dx}(x^p) = px^{p-1} \tag{14}$$

Furthermore, if u is a positive differentiable function on its domain, then

$$\frac{d}{dx}(u(x)^p) = p(u(x))^{p-1} \times u'(x) \tag{15}$$

Derivative of $\log_b x$

If $b > 1$,

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b} \quad \forall x \in \mathbf{R}^+ \wedge x \neq 0 \tag{16}$$

$$\frac{d}{dx}(\log_b |x|) = \frac{1}{x \ln b} \quad \forall x \in \mathbf{R}^+ \wedge x \neq 0 \tag{17}$$

7.4 Inverse Trigonometric Functions

Derivative of Inverse Trigonometric Functions

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\quad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \{x \in \mathbf{R} \,|\, -1 < x < 1\}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\quad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, \{x \in \mathbf{R} \,|\, -\infty < x < \infty\}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$ $\quad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}, \{x \in \mathbf{R} \,|\, |x| > 1\}$

Integrals Involving Inverse Trigonometric Functions

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C \tag{18}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C \tag{19}$$

$$\int \frac{dx}{x\sqrt{a^2 + x^2}} = \frac{1}{a} \sec^{-1} \left|\frac{x}{a}\right| + C \tag{20}$$

7.5 L'Hôpital's Rule and Growth Rates of Functions

Indeterminate forms $1^\infty, 0^0, \infty^0$

Assume $\lim_{num_1} f(x)^{g(x)}$ has the indeterminate form 1^∞, 0^0, or ∞^0.

1. Evaluate $L = \lim_{num_1} g(x) \ln f(x)$. This limit can be put in the form $0/0$ or ∞/∞, both of which are handled by l'Hôpital's rule.

2. Then $\lim_{num_1} f(x)^{g(x)} = e^L$

Growth Rates of Functions (as $x \to \infty$)

Suppose f and g are functions with $\lim_{num_1} f(x) = \lim_{num_1} g(x) = \infty$. Then f **grows faster than g** as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \quad \text{or, quantitatively, if} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \tag{21}$$

The functions f and g have **comparable growth rates** if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M$$

where $M \in \mathbb{R}^+$.

Ranking Growth Rates as $x \to \infty$

Let $f << g$ mean that g grows faster than f as $f \to \infty$. With positive real numbers p, q, r, s and $b > 1$,

$$\ln^q x << x^p << x^p \ln^r x << x^{p+s} << b^x << x^x \tag{22}$$

7.6 Hyperbolic Functions

Hyperbolic Functions

Hyperbolic Cosine

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{23}$$

Hyperbolic Sine

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{24}$$

Hyperbolic Tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{25}$$

Hyperbolic Cotangent

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \tag{26}$$

Hyperbolic Secant

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \tag{27}$$

Hyperbolic Cosecant

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \tag{28}$$

Hyperbolic Identities

$\cosh^2 x - \sinh^2 x = 1$	$\cosh(-x) = \cosh x$
$1 - \tanh^2 x = \operatorname{sech}^2 x$	$\sinh(-x) = -\sinh x$
$\coth^2 x - 1 = \operatorname{csch}^2 x$	$\tanh(-x) = -\tanh x$
$\coth(x + y) = \cosh x \cosh y + \sinh x \sinh y$	
$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$	
$\cosh 2x = \cosh^x + \sinh^2$	$\sinh 2x = 2 \sinh x \cosh x$
$\cosh^2 x = \frac{\cosh 2x+1}{2}$	$\sinh^2 x = \frac{\cosh 2x-1}{2}$

Derivatives and Integral Formulas

1. $\frac{d}{dx}(\cosh x) = \sinh x \quad \Rightarrow \quad \int \sinh x \; dx = \cosh x + C$
2. $\frac{d}{dx}(\sinh x) = \cosh x \quad \Rightarrow \quad \int \cosh x \; dx = \sinh x + C$
3. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \quad \Rightarrow \quad \int \operatorname{sech}^2 x \; dx = \tanh x + C$
4. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x \quad \Rightarrow \quad \int \operatorname{csch}^2 x \; dx = -\coth x + C$
5. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \quad \Rightarrow \quad \int \operatorname{sech} x \tanh x \; dx = -\operatorname{sech} x + C$
6. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x \quad \Rightarrow \quad \int \operatorname{csch} x \coth x \; dx = -\operatorname{csch} x + C$

Integrals of Hyperbolic Functions

1. $\int \tanh x \, dx = \ln \cosh x + C$
2. $\int \coth x \, dx = \ln |\sinh x| + C$
3. $\int \operatorname{sech} x \, dx = \tan^{-1} |\sinh x| + C$
4. $\int \operatorname{csch} x \, dx = \ln |\tanh(x/2)| + C$

Inverses of the Hyperbolic Functions Expressed as Logarithms

$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \ (x \geq 1)$ $\quad \operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x} \ (0 < x \leq 1)$
$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$ $\quad \operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x} \ (x \neq 0)$
$\tanh^{-1} x = \frac{1}{2} \ln(\frac{1+x}{1-x}) \ (|x| < 1)$ $\quad \coth^{-1} x = \tanh^{-1} \frac{1}{x} \ (|x| > 1)$

Derivatives of the Inverse Hyperbolic Functions

$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}} \ (x > 1)$ $\quad \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$
$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \ (|x| < 1)$ $\quad \frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2} \ (|x| > 1)$
$\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}} \ (0 < x < 1)$ $\quad \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{1+x^2}} \ (x \neq 0)$

Integral Formulas

1. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1} \frac{x}{a} + C, \text{ for } x > a$
2. $\int \frac{dx}{x^2+a^2} = \sinh^{-1} \frac{x}{a} + C, \text{ for all } x$

3. $\int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C,\ \text{for}|x| < a = \frac{1}{a}\coth^{-1}\frac{x}{a} + C, \text{for } |x| > a$

4. $\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\frac{x}{a} + C,\ \text{for } 0 < x < a$

5. $\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\frac{|x|}{a} + C,\ \text{for } x \neq 0$

8 Integration Techniques

8.1 Basic Approaches

$$\int k\,dx = kx + C \tag{29}$$

$$\int k^p\,dx = \frac{k^{p+1}}{p+1} + C, p \in \mathbf{R} \wedge \neq -1 \tag{30}$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C \tag{31}$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C \tag{32}$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C \tag{33}$$

$$\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C \tag{34}$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C \tag{35}$$

$$\int \csc ax \cot ax\,dx = -\frac{1}{a}\csc ax \cot ax + C \tag{36}$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C \tag{37}$$

$$\int \frac{dx}{x} = \ln|x| + C \tag{38}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C \tag{39}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \tag{40}$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C \tag{41}$$

8.2 Integration By Parts

Suppose that u and v are differentiable functions. Then

$$\int u\,dv = uv - \int v\,du \tag{42}$$

Integration by Parts for Definite Integrals

Let u and v be differentiable. Then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx \tag{43}$$

Integral of $\ln x$

$$\int \ln x\,dx = x\ln x - x + C \tag{44}$$

8.3 Trigonometric Integrals

$\int \sin^m x \cos^n x \; dx$ **Strategy**

m **is odd,** n **real** Split off $\sin x$, rewrite the resulting even power of $\sin x$ in terms of $\cos x$, and then use $u = \cos x$

n **odd,** m **real** Split off $\cos x$, rewrite the resulting even power of $\cos x$ in terms of $\sin x$, and then use $u = \sin x$.

m **and** n **both even, nonnegative** Use half-angle identities to transform the integrand into a polynomial in $\cos 2x$, and apply the preceding strategies once again to powers of $\cos 2x$ greater than 1.

Reduction Formulas

Assume n is a positive integer.

$$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \; dx \tag{45}$$

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \; dx \tag{46}$$

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \; dx, n \neq 1 \tag{47}$$

$$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \; dx, n \neq 1 \tag{48}$$

Integrals of $\tan x, \cot x, \sec x,$ and $\csc x$

$$\int \tan x \, dx = -\ln|\cos x| + C = \ln|\sec x| + C \tag{49}$$

$$\int \cot x \, dx = \ln|\sin x| + C \tag{50}$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C \tag{51}$$

$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C \tag{52}$$

$\int \tan^m x \sec^n x \, dx$ **Strategy**

n **even** Split off $\sec^2 x$, rewrite the remaining even power of $\sec x$ in terms of $\tan x$, and use $u = \tan x$.

m **odd** Split off $\sec x \tan x$, rewrite the remaining even power of $\tan x$ in terms of $\sec x$, and use $u = \sec x$.

m **even and** n **odd** Rewrite the even power of $\tan x$ in terms of $\sec x$ to produce a polynomial in $\sec x$; apply reduction formula 4 to each term.

8.4 Trigonometric Substitutions

The Integral Contains...	**Corresponding Substitution**	**Useful Identity**
$a^2 - x^2$	$x = a\sin\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \forall \lvert x\rvert \leq a$	$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$a^2 + x^2$	$x = a\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$x^2 - a^2$	$x = a\sec\theta,$ $\begin{cases} 0 \leq \theta < \frac{\pi}{2}, \forall x \geq a \\ \frac{\pi}{2} < \theta \leq \pi, \forall x \leq -a \end{cases}$	$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

8.5 Partial Fractions

Partial Fractions with Simple Linear Factors

Suppose $f(x) = p(x) > q(x)$, where p and q are polynomials with no common factors and with the degree of p less than the degree of q. Assume that q is the product of simple linear factors. The partial fraction decomposition is obtained as follows.

Step 1. Factor the denominator q in the form $(x-r_1)(x-r_2)\cdots(x-r_n)$, where $r_1, \ldots, r_n$ are real numbers.

Step 2. Partial fraction decomposition Form the partial fraction decomposition by writing

$$\frac{p(x)}{q(x)} = \frac{A_1}{x-r_1} + \frac{A_2}{x-r_2} + \cdots + \frac{A_n}{x-r_n} \tag{53}$$

Step 3. Clear denominators Multiply both sides of the equation in Step 2 by $q(x) = (x-r_1)(x-r_2)\cdots(x-r_n)$, which produces conditions for $A_1, \ldots, A_n$.

Step 4. Solve for coefficients Equate like powers of x in Step 3 to solve for the undertmined coefficients $A_1, \ldots, A_n$

Partial Fractions For Repeated Linear Factors

Suppose the repeated linear factor $(x-r)^m$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition has a partial fraction for each power of $(x-r)$ up to and including the mth power; that is, the partial fraction decomposition contains the sum

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m} \tag{54}$$

where $A_1, \ldots, A_m$ are constants to be determined.

Partial Fractions with Simple Irreducible Quadratic Factors

Suppose a simple irreducible factor $ax^2 + bx + c$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition contains a term of the form

$$\frac{Ax + B}{ax^2 + bx + c} \tag{55}$$

where A and B are unknown coefficients to be determined.

Partial Fraction Decomposition

Let $f(x) = p(x)/q(x)$ be a proper rational function in reduced form. Assume the denominator q has been factored completely over the real numbers and m is a positive integer.

Simple linear factor A factor $x - r$ in the denominator requires the partial fraction $\frac{A}{x-r}$.

Repeated linear factor A factor $(x - r)^m$ with $m > 1$ in the denominator requires the partial fractions.

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m} \tag{56}$$

Simple irreducible quadratic factor An irreducible factor $ax^2 + bx + c$ in the denominator requires the partial fraction

$$\frac{Ax + B}{ax^2 + bx + c} \tag{57}$$

Repeated irreducible quadratic factor An irreducible factor $(ax^2 + bx + c)^m$ with $m > 1$ in the denominator requires the partial fractions

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m} \tag{58}$$

8.6 Improper Integrals

Improper Integrals over Infinite Intervals

1. If f is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx \tag{59}$$

provided the limit exists.

2. If f is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx \tag{60}$$

provided the limit exists.

3. If f is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx \tag{61}$$

provided both limits exist, where c is any real number.

In each case, if the limit exists, the improper integral is said to **converge**, if it does not exist, the improper integral is said to **diverge**.

Improper Integrals with an Unbounded Integrand

1. Suppose f is continuous on $(a, b]$ with $\lim_{{}_n um_1} f(x) = \pm\,\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx \tag{62}$$

provided the limit exists.

2. Suppose f is continuous on $[a, b)$ with $\lim_{{}_n um_1} f(x) = \pm\,\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx \tag{63}$$

provided the limit exists.

3. Suppose f is continuous on $[a, b]$ except at the interior point p where f is unbounded. Then

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx, \tag{64}$$

provided the improper integrals on the right side exist.

In each case, if the limit exists, the improper integrals is said to **converge**, if it does not exists, the improper integral is said to **diverge**.

9 Sequences and Infinite Series

9.1 An Overview

Sequence

A **sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\} \tag{65}$$

A sequence may be generated by a **recurrence relations** of the form $a_{n+1} = f(a_n)$, for $n = 1, 2, 3, \ldots$, where a_1 is given. A sequence may also be defined with an **explicit form** of the form $a_n = f(n)$, for $n = 1, 2, 3, \ldots$.

Limit of a Sequence

If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases, then we say $\lim\limits_{{}_n um_1} = L$ exists, and the sequence **converges** to L. If the terms of the sequence do not approach a single number as n increases, the sequence has no limits, and the sequence **diverges**.

Infinite Series

Given a set of numbers $\{a_1, a_2, a_3, \ldots\}$, the sum

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k \tag{66}$$

is called an **infinity series**. Its **sequence of partial sums** $\{S_n\}$ has the terms

$$S_1 = a_1 \tag{67}$$

$$S_2 = a_1 + a_2 \tag{68}$$

$$S_3 = a_2 + a_2 + a_3 \tag{69}$$

$$\vdots \tag{70}$$

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \quad n = 1, 2, 3, \ldots \tag{71}$$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series **converges** to that limits, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L \tag{72}$$

If the sequence of partial sums diverges, the infinite series also **diverges**.

9.2 Sequences

Limits of Sequences from Limits of Functions

Suppose f is a function such that $f(n) = a_n$ for all positive integers n. If $\lim_{num_1} = L$, then the limits of the sequences $\{a_n\}$ is also L.

Properties of Limits of Sequences

Assume that the sequence $\{a_n\}$ and $\{b_n\}$ have limits A and B, respectively. Then,

1. $\lim_{num_1} (a_n \pm b_n) = A \pm B$
2. $\lim_{num_1} ca_n = cA$, where c is a real number
3. $\lim_{num_1} a_n b_n = AB$
4. $\lim_{num_1} \frac{a_n}{b_n} = \frac{A}{B}$, provided $B \neq 0$.

Geometric Sequences

Let r be a real number. Then,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \Leftrightarrow |r| < 1 \\ 1 & \Leftrightarrow r = 1 \\ \text{does not exist} & \Leftrightarrow r \leq -1 \vee r > 1 \end{cases} \tag{73}$$

Squeeze Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences with $a_n \leq b_n \leq c_n$ for all n greater than some index N. If $\lim_{num_1} a_n = \lim_{num_1} c_n = L$, the $\lim_{num_1} b_n = L$.

Bounded Monotonic Sequences

A bounded monotonic sequence converges.

Growth Rates of Sequences

The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$

$$\{\ln^q n\} \ll \{n^p\} \ll \{n^p \ln^r n\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\} \tag{74}$$

The ordering applies for $p, q, r, s, b \in \mathbb{R}^+ \wedge b > 1$.

Limit of a Sequence

The sequence $\{a_n\}$ converges to L provided the terms of a_n can be made arbitrarily close to L by taking n sufficiently large. More precisely, $\{a_n\}$ has the unique limit L if given any tolerance $\epsilon > 0$, it is possible to find a positive integer N (depending only on ϵ) such that

$$|a_n - L| < \epsilon \quad \text{whenever } n > N \tag{75}$$

if the **limit of a sequence** is L, we say the sequence **converges** to L, written

$$\lim_{n\to\infty} a_n = L \tag{76}$$

A sequence that does not converge is said to **diverge.**

9.3 Infinite Series

Geometric Series

$$\sum_{k=0}^{n-1} ar^k = S_n = a\frac{1-r^n}{1-r} \tag{77}$$

Geometric Series

Let $a \neq 0$ and r be real numbers. If $|r| < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. If $|r| \geq 1$, then the series diverges.

9.4 The Divergence and Integral Tests

Divergence Test

If $\sum a_k$ converges, then $\lim\limits_{num_1} a_k = 0$. Equivalently, if $\lim\limits_{num_1} a_k \neq 0$, then the series diverges. However, this cannot be used to prove convergence. If $\lim\limits_{num_1} a_k = 0$, the test is inconclusive.

Harmonic Series

The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ diverges, even though the terms of the series approach zero.

Integral Test

Suppose f is a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Then

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \int_1^{\infty} f(x)\,dx \tag{78}$$

either both converge or both diverge. In the case of convergence, the value of the integral is *not*, in general, equal to the value of the series.

Convergence of the p-Series

The p-Series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges, for $p > 1$, and diverges for $p \leq 1$.

Estimating Series with Positive Terms

Let f be continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Let $S = \sum_{k=1}^{\infty} a_k$ be a convergence series and let $S_n = \sum_{k=1}^{n}$ be the sum of of the first n terms of the series. The remainder $R_n = S - S_n$ satisfies

$$R_n \leq \int_n^{\infty} f(x)\,dx \tag{79}$$

Furthermore, the exact value of the series is bounded as follows:

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \leq \sum_{k=1}^{\infty} a_k \leq S_n + \int_n^{\infty} f(x)\,dx. \tag{80}$$

Properties of Convergent Series

1. Suppose $\sum a_k$ converges to A and let c be a real number. The series $\sum ca_k$ converges and $\sum ca_k = c\sum a_k = cA$

2. Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B. The series $\sum(a_k \pm b_k)$ converges and $\sum(a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$

3. *Whether* a series converges does not depend on a fininte number of terms added to or removed form the series. Specifically, if M is a positive integer, then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=M}^{\infty} a_k$ both converge or both diverge. However, the *value* of a convergent series does change if nonzero terms are added or deleted.

9.5 The Ratio, Root, and Comparison Tests

Useful Identities

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^k = e \tag{81}$$

$$\lim_{k\to\infty} k^{\frac{1}{k}} = 1 \tag{82}$$

The Ratio Test

Let $\sum a_k$ be an infinite series with positive terms and let $r = \lim_{num_1} \frac{a_{k+1}}{a_k}$.

1. If $0 \le r < 1$, the series converges.
2. If $r > 1$ (including $r = \infty$), the series diverges.
3. If $r = 1$, the test is inconclusive.

The Root Test

Let $\sum a_k$ be an infinite series with nonegative terms and let $\rho = \lim_{num_1} \sqrt[k]{a_k}$.

1. If $0 \le \rho < 1$, the series converges.
2. If $\rho > 1$ (including $\rho = \infty$), the series diverges.
3. If $\rho = 1$, test is inconclusive.

The Comparison Test

Let $\sum a_k$ and $\sum b_k$ be a series with postivie terms.

1. If $0 < a_k \le b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges.
2. If $0 < b_k \le a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

The Limit Comparison Test

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L \tag{83}$$

- If $0 < L < \infty$ (that is, L is a finite, positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.
- If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.
- If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Guidelines

- Begin with the Divergence Test. If you show that $\lim_{num_1} a_k \neq 0$, then the series diverges and your work is finished.
- Is the series a special series? Recall the convergence properties for the following series:
  - Geometric series: $\sum ar^k$ converges for $|r| < 1$ and diverges for $|r| \geq 1$ $(a \neq 0)$.
  - p-series: $\sum \frac{1}{k^p}$ converges for $p > 1$, and diverges for $p \leq 1$.
  - Check also for telescoping series.
- If the general kth term of the series looks like a function you can integrate, then try the Integral Test.
- If the general kth term of the series involves $k!, k^k,$ or a^k, where a is a constant, the Ratio Test is advisable. Series with k in an exponent may yield to the Root Test.
- If the general kth term of the series is a rational function of k (or a root of a rational function), use the Comparison or the Limit Comparison Test. Use the families of series given in Step 2 as comparison series.

9.6 Alternating Series

The Alternating Series Test

The alternating series $\sum(-1)^{k+1}a_k$ converges provided

1. the terms of the series are non-increasing in magnitude ($0 < a_{k+1} \leq a_k$, for k greater than some index N) and
2. $\lim\limits_{num_1} = 0$

Alternating Harmonic Series

The alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ converges (even though the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ diverges).

Remainder in Alternating Series

Let $R_n = |S - S_n|$ be the remainder in approximating the value of a convergent alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ by the sum of its first n terms. Then $R_n \leq a_{n+1}$. In other words, the remainder is less than or equal to the magnitude of the first neglected term.

Absolute and Conditional Convergence

Assume the infinite series $\sum a_k$ converges. The series $\sum a_k$ **converges absolutely** if the series $\sum |a_k|$ converges. Otherwise, the series $\sum a_k$ **converges conditionally**.

Absolute Convergence Implies Convergence

If $\sum |a_k|$ converges, then $\sum a_k$ converges (absolute convergence implies convergence). If $\sum a_k$ diverges, then $\sum |a_k|$ diverges.

Series or Test	**Form of Series**	**Condition for Convergence**	**Condition for Divergence**	**Comments**
Geometric Series	$\sum_{k=0}^{\infty} ar^k, a \neq 0$	$\|r\| < 1$	$\|r\| \geq 1$	If $\|r\| < 1$, then $\sum a_{k=1}^{\infty} ar^k = \frac{a}{1-r}$
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does Not Apply	$\lim_{k\to\infty a_k \neq 0}$	Cannot be used to prove convergence.
Integral Test	$\sum a_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing.	$\int_1^{\infty} f(x)dx < \infty$	$\int_1^{\infty} f(x)dx$ does not exist	The value of the integral is not the value of the series.
p-Series	$\sum a_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests.
Ratio Test	$\sum a_{k=1}^{\infty}$ where $a_k > 0$	$\lim_k \to \infty \frac{a_{k+1}}{a_k} < 1$	$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} > 1$	Inconclusive if $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$
Root Test	$\sum a_{k=1}^{\infty}$ where $a_k \geq 0$	$\lim_{k\to\infty} \sqrt[k]{a_k} < 1$	$\lim_{k\to\infty} \sqrt[k]{a_k} > 1$	Inconclusive if $\lim_{k\to\infty} \sqrt[k]{a_k} = 1$
Comparison Test	$\sum a_{k=1}^{\infty}$ where $a_k > 0$	$0 < a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges	$0 < b_k \leq a_k$ and $\sum a_{k=1}^{\infty} b_k$ diverges	$\sum a_{k=1}^{\infty} a_k$ is given; you supply $\sum a_{k=1}^{\infty} b_k$
Limit Comparison Test	$\sum a_{k=1}^{\infty} a_k$ where $a_k > 0, b_k > 0$	$0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges	$\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum a_{k=1}^{\infty} b_k$ diverges.	$\sum a_{k=1}^{\infty} a_k$ is given; you supply $\sum a_{k=1}^{\infty} b_k$
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0, 0 < a_{k+1} \leq a_k$	$\lim_{k\to\infty} a_k = 0$	$\lim_{k\to\infty} a_k \neq 0$	Remainder R_n satisfies $R_n \leq a_{n+1}$
Absolute Convergence	$\sum a_{k=1}^{\infty} a_k, a_k$ arbitrary	$\sum_{k=1}^{\infty} \|a_k\|$ converges.	Applies to arbitrary series	

10 Power Series

10.1 Approximating Functions with Polynomials

Taylor Polynomials

Let f be a function with $f', f'', \ldots, f^{(n)}$ defined at a. The n**th-order Taylor polynomial** for f with its **center** at a, denoted p_n, has the property that it matches f in value, slope, and all derivatives up to the nth derivative at a; that is,

$$p_n(a) = f(a), p_n'(a) = f'(a), \ldots, p_n^{(n)}(a) = f^{(n)}(a). \tag{84}$$

The nth-order Taylor polynomial centered at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{85}$$

More compactly, $p_n(x) = \sum_{k=0}^{n} c_k(x-a)^k$, where the **coefficients** are

$$c_k = \frac{f^{(k)}(a)}{k!}, \quad \text{for } k = 0, 1, 2, \ldots, n \tag{86}$$

Remainder in a Taylor Polynomial

Let p_n be the Taylor polynomial of order n for f. The **remainder** in using p_n to approximate f at the point x is

$$R_n(x) = f(x) - p_n(x) \tag{87}$$

Taylor's Theorem

Let f have continuous derivatives up to $f^{(n+1)}$ on an open interval I containing a. For all x in I,

$$f(x) = p_n(x) + R_n(x), \tag{88}$$

where p_n is the nth-order Taylor polynomial for f centered at a, and the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{(n+1)}, \tag{89}$$

for some point c between x and a.

Estimate of the Remainder

Let n be a fixed positive integer. Suppose there exists a number M such that $|f^{(n+1)}(c)| \leq M$, for all c between a and x inclusive. The remainder in the nth-order Taylor polynomial for f centered at a satisfies

$$|R_n(x)| = |f(x) - p_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!} \tag{90}$$

10.2 Properties of Power Series

Power Series

A **power series** has the general form

$$\sum_{k=0}^{\infty} c_k(x-a)^k \tag{91}$$

where a and c_k are real numbers, and x is a variable. The c_k's are the **coefficients** of the power series and a is the **center** of the power series. The set of values of x for which the series converges is its **interval of convergence**. The **radius of convergence** of the power series, denoted R, is the distance from the center of the series to the boundary of the interval of convergence.

Convergence of Power Series

A power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ centered at a converges in one of three ways:

1. The series converges absolutely for all x, in which case the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.

2. There is a real number $R > 0$ such that the series converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$, in which case the radius of convergence is $R = 0$.

3. The series converges only at a, in which case the radius of convergence is $R = 0$.

Combining Power Series

Suppose the power series $\sum c_k x^k$ and $\sum d_k x^k$ converge absolutely to $f(x)$ and $g(x)$, respectively, on an interval I.

1. **Sum and difference:** The power series $\sum (c_k \pm d_k)x^k$ converges absolutely to $f(x) \pm g(x)$ on I.

2. **Multiplication by a power:** The power series $x^m \sum c_k x^k = \sum c_k x^{k+m}$ converges absolutely to $x^m f(x)$ on I, provided m is an integer such that $k + m \geq 0$ for all terms of the series.

3. **Composition:** If $h(x) = bx^m$, where m is a positive integer and b is a real number, the power series $\sum c_k(h(x))^k$ converges absolutely to the composite function $f(h(x))$, for all x such that $h(x)$ is in I.

Differentiating and Integrating Power Series

Let the function f be defined by the power series $\sum c_k(x-a)^k$ on its interval of convergence I.

- f is a continuous function on I.
- The power series may be differentiated or integrated term by term, and the resulting power series converges to $f'(x)$ or $\int f(x)\,dx + C$, respectively, at all points in the interior of I, where C is an arbitrary constant.

10.3 Taylor Series

Taylor/Maclaurin Series for a Function

Suppose the function f has derivatives of all orders on an interval containing the point a. The **Taylor series for f centered at a** is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots \tag{92}$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{93}$$

A Taylor series centered at 0 is called a **Maclaurin series**.

Binomial Coefficients

$\forall p, k \in \mathbb{R} \wedge k \geq 1$

$$\binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}. \tag{94}$$

With the special case of $\binom{p}{0} = 1$.

Binomial Series

$\forall p \in \mathbb{R} \wedge p \neq 0$, the Taylor series for $f(x) = (1+x)^p$ centered at 0 is the **binomial series**

$$\sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!} x^k \tag{95}$$

$$= 1 + px + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \tag{96}$$

The series converges for $|x| < 1$ (and possibly at the endpoints, depending on p). If p is a nonnegative integer, the series terminates and results in a polynomial of degree p.

Convergence of Taylor Series

Let f have derivatives of all orders on an open interval I containing a. The Taylor series for f centered at a converges to f , for all x in I, if and only if $\lim_{n\to\infty} R_n(x) = 0$, for all x in I, where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \tag{97}$$

is the remainder at x (with c between x and a).

Taylor Series Functions

11 Parametric and Polar Curves

11.1 Parametric Equations

Forward or Positive Orientation

The direction in which a parametric curve is generated as the parameter increases is called the **forward**, or **positive**, **orientation** of the curve.

Derivative for Parametric Curves

Let $x = g(t)$ and $y = h(t)$, where g and h are differentiable on an interval $[a, b]$. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} \qquad (98)$$

provided $\frac{dx}{dt} \neq 0$.

11.2 Polar Coordinates

Introduction

- Up to now we have only studied in a Cartesian coordinate system.
  - A Cartesian coordinate system is just a plane described by Cartesian (or, algebraic) equations and points in a finite dimensions.
    - *One Dimension*: Lines.
    - *Two Dimensions*: x^2.
    - *Three, Four*: Upper-level Calculus and Physics.
- Let's define an alternative coordinate system — **polar coordinate.**
  - coordinates are constants on circles and rays.
  - Useful for navigation, position, and gravitation fields.

Defining Polar Coordinates

Pole The origin of the coordinate system.

Polar Axis Synonymous for the positive x-axis.

Polar Coordinates A polar coordinates P has the form (r, θ).

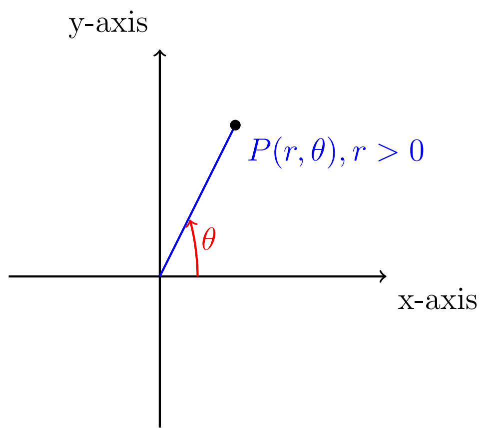
Radial Coordinate The radial coordinate r describes the *signed*, or *directed*, distance from the origin to P.

Angular Coordinate The angular coordinate θ describes an angle whose initial side is the positive x-axis and whose terminal side lies on the ray passing through the origin and P.

Notes

- **Positive angle measurements are measured *counterclockwise* from the origin.**
- Every point has multiple representations.
  - Angles are periodic, so multiples of 2π gives the same angle.

- Coordinates may be negative. So (r, θ) can be represented as $(-r, \theta + \pi)$ and $(-r, \theta - \pi)$



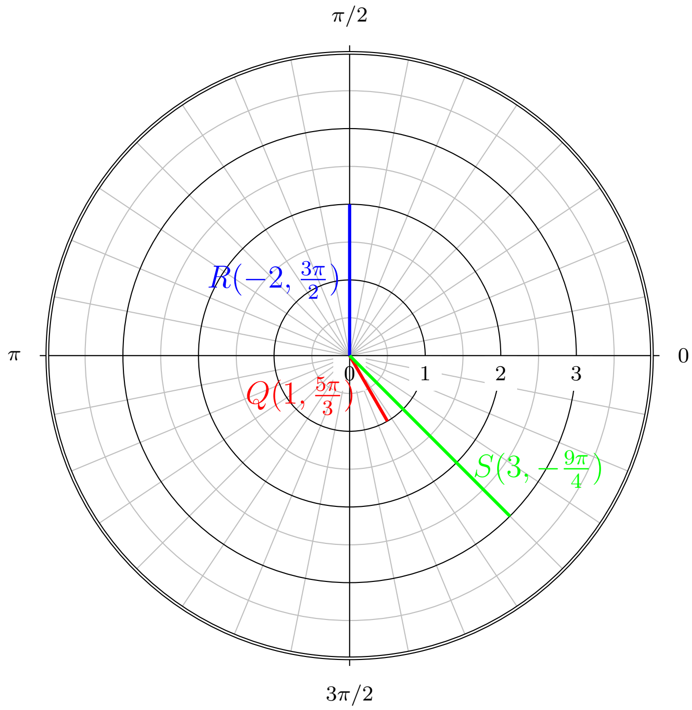
In summary,

- $(r, \theta + 2\pi)$ represents the same point as (r, θ)
- $P(r, \theta)$ and $P'(-r, \theta)$ are reflections through the origin.

Examples

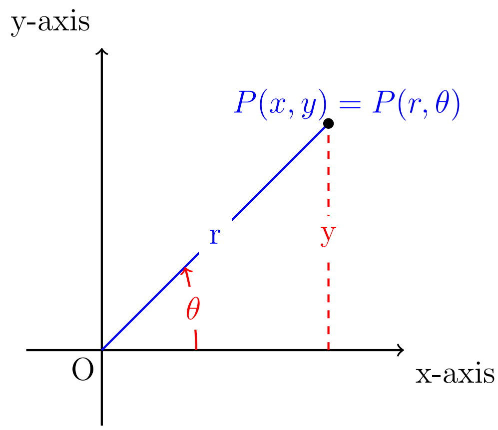
Graph the following points in polar coordinates:

- $Q(1, \frac{5\pi}{3})$
- $R(-2, \frac{3\pi}{2})$
- $S(3, -\frac{9\pi}{4})$
    - Now give two alternative representations.
    - $S'(3, \frac{1\pi}{4})$
    - $S''(-3, -\frac{5\pi}{4})$



Converting Between Cartesian and Polar Coordinates

- We sometimes need to convert between Cartesian and polar coordinates.
- Let's turn this problem into a right triangle.



A point with polar coordinates (r, θ) has Cartesian coordinates (x, y), where

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \tag{99}$$

A point with Cartesian coordinates (x, y) has polar coordinates (r, θ), where

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \tag{100}$$

Examples

BE SURE TO GRAPH POINTS IN CARTESIAN FIRST.

Polar to Cartesian

Express the point with the following polar coordinates in Cartesian coordinates: $P(3, \frac{2\pi}{3})$

$$\begin{aligned} x &= r\cos\theta \\ &= 3\cos(^{2\pi}/_{3}) \\ &= -3(^{1}/_{2}) \\ &= -^{3}/_{2} \end{aligned} \qquad \begin{aligned} y &= r\sin\theta \\ &= 3\sin(^{2\pi}/_{3}) \\ &= 3(^{\sqrt{3}}/_{2}) \\ &= ^{3\sqrt{3}}/_{2} \end{aligned}$$

Polar to Cartesian

Express the point with the following polar coordinates in Cartesian coordinates: $Q(e, -\frac{\pi}{4})$

$$\begin{aligned} x &= r\cos\theta \\ &= e\cos(-^{\pi}/_{4}) \\ &= e(^{\sqrt{2}}/_{2}) \\ &= ^{e\sqrt{2}}/_{2} \end{aligned} \qquad \begin{aligned} y &= r\sin\theta \\ &= e\sin(-^{\pi}/_{4}) \\ &= -e(^{\sqrt{2}}/_{2}) \\ &= -^{e\sqrt{2}}/_{2} \end{aligned}$$

Cartesian To Polar

Express the point with the following Polar coordinates to Cartesian coordinates: $R(1, -1)$

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{1^2+(-1)^2} \\ &= \sqrt{2} \end{aligned} \qquad \begin{aligned} \tan\theta &= {}^{y}\!/_{x} \\ &= {}^{-1}\!/_{1} \\ &= -1 \\ \theta &= {}^{-\pi}\!/_{4} \text{ or } {}^{7\pi}\!/_{4}. \end{aligned}$$

Therefore, two possible solutions are: $(\sqrt{2}, {}^{-\pi}\!/_{4})$ or $(\sqrt{2}, {}^{7\pi}\!/_{4})$

Cartesian To Polar

Express the point with the following Polar coordinates to Cartesian coordinates: $S(1, \sqrt{3})$

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{(\sqrt{3})^2+(1)^2} \\ &= 2 \end{aligned} \qquad \begin{aligned} \tan\theta &= {}^{y}\!/_{x} \\ &= {}^{\sqrt{(3)}}\!/_{1} \\ \theta &= {}^{\pi}\!/_{3} \text{ or } {}^{4\pi}\!/_{3}. \end{aligned}$$

Therefore, two possible solutions are: $(2, {}^{\pi}\!/_{3})$ or $(2, {}^{4\pi}\!/_{3})$.

Basic Curves in Polar Coordinates

- A curve in polar coordinates is the set of **points** that satisfy an equation in r and θ.
- This makes graphing some things easier than others.
- Look at $r = 3$ is the set of all points that satisfy being away from the origin of 3 units.
  - This is because θ is not specified, it's arbitrary. Basically, θ is the function.
  - In general, $r = a$, $\forall a \in \mathbb{R}^+$ describes a circle.
- Taking the converse, let r be arbitrary.
  - If the r is arbitrary, and we specify the angle, what do you think we get?

- A line!
- Take $\sqrt{2}/2$.

Polar to Cartesian Graph Example

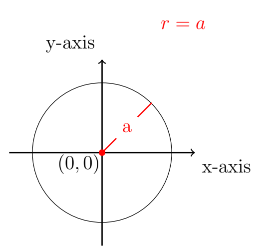
Convert the polar equation $r = 6\sin\theta$ to Cartesian coordinates and describe the corresponding graph.

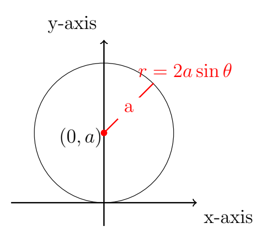
$$r^2 = 6r\sin\theta \tag{101}$$
$$x^2 + y^2 = 6y \tag{102}$$
$$0 = x^2 + y^2 - 6y \tag{103}$$
$$= x^2 + (y^2 - 6y + 9) - 9 \tag{104}$$
$$= x^2 + (y - 3)^2 - 9 \tag{105}$$

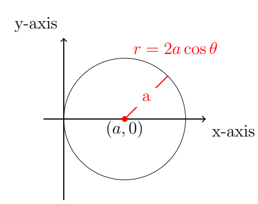
We recognize this to be the equation of a circle, centered at $(0, 3)$ at 3. We can also generalize this.

Circle in Polar Coordinates

- The equation $r = a$ describes a circle of radius $|a|$ centered at $(0, 0)$.
- The equation $r = 2a\sin\theta$ describes a circle of radius $|a|$ centered at $(0, a)$.
- The equation $r = 2a\cos\theta$ describes a circle of radius $|a|$ centered at $(a, 0)$.



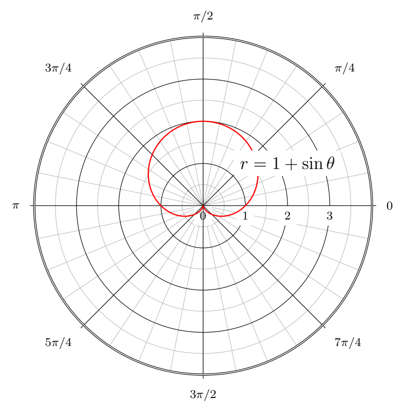




Graphing In Polar Coordinates

Graph the polar equation $r = f(\theta) = 1 + \sin\theta$

θ	$r = 1 + sin\theta$
0	1
$\pi/6$	$3/2$
$\pi/2$	2
$5\pi/6$	$3/2$
π	1
$7\pi/6$	$1/2$
$3\pi/2$	0
$11\pi/6$	$1/2$
2π	1



The resulting curve is known as a **cardioid**.

Cartesian-to-Polar Method for Graphing $r = f(\theta)$

1. Graph $r = f(\theta)$ *as if* r *and* θ *were Cartesian coordinates* with θ on the horizontal axis and r on the vertical axis. Be sure to choose an interval in θ on which the entire polar curve is produced.

2. Use the Cartesian graph in Step 1 as a guide to sketch the points (r, θ) on the final *polar* curve.

Example

With the alternate graphing method, graph $r = 1 + \sin\theta$.

Symmetry In Polar Equations

Symmetry about the x-axis occurs if the point (r, θ) is on the graph whenever $(r, -\theta)$ is on the graph.

Symmetry about the y-axis occurs if the point (r, θ) is on the graph whenever $(r, \pi - \theta) = (-r, -\theta)$ is on the graph.

Symmetry about the origin occurs if the point $(r, \theta$ is on the graph whenever $(-r, \theta) = (r, \theta + \pi)$ is on the graph.

11.3 Calculus in Polar Coordinates

Slope of a Tangent Line

Let f be a differentiable function at θ_0. The slope of the line tangent to the curve $r = f(\theta)$ at the point $(f(\theta_0, \theta_0)$ is

$$\frac{dy}{dx} = \frac{f'(\theta_0)\sin\theta_0 + f(\theta_0)\cos\theta_0}{f'(\theta_0)\cos\theta_0 - f(\theta_0)} \tag{106}$$

Area of Regions in Polar Coordinates

Let R be the region bounded by the graphs of $r = f(\theta)$ and $r = g(\theta)$, between $\theta = \alpha$ and $\theta = \beta$, where f and g are continous and $f(\theta) \geq g(\theta) \geq 0$ on $[\alpha, \beta]$. The area of R is

$$\int_{\alpha}^{\beta} \frac{1}{2}(f(\theta)^2 - g(\theta)^2)\, d\theta \tag{107}$$

MATH2100

Foundations Of Mathematics

Chapter _num$_1$

- We talked about using rational numbers as a model for our axiom systems.
  - Rationals work, upto Axiom 6.
  - Suppose we take π as the dividing step. The hypothesis is now satisfied, but either S_1 would not have a top level or S_2 would not have a bottom level.

MATH2222

Calculus III

Chapter 12: Vectors and Vector Valued Functions

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Contents

12.1 Vectors in the Plane

Vectors, Equal Vectors, Scalars, Zero Vector

Vectors are quantities that have both **length** (or **magnitude**) and **direction**. Two vectors are **equal** if they have the same magnitude and direction. Quantities having magnitude but no direction are called **scalars**. One exception is the **zero** vector, denoted $\mathbf{0}$: It has length 0 and no direction.

Scalar Multiples and Parallel Vectors

Given a scalar c and a vector $\mathbf{u}$, the scalar multiple $c\mathbf{v}$ is a vector whose magnitude is $|c|$ multiplied by the magnitude of $\mathbf{v}$. If $c > 0$, then $c\mathbf{v}$ has the same direction as $\mathbf{v}$. If $c < 0$, then $c\mathbf{v}$ and $\mathbf{v}$ point in opposite directions. Two vectors are **parallel** if they are scalar multiples of each other.

Position Vectors and Vector Components

A vector $\mathbf{v}$ with its tail at the origin and head at the point (v_1, v_2) is called a **position vector** (or is said to be in **standard position**) and is written $\langle v_1, v_2\rangle$. The real numbers v_1 and v_2 are the x- and y-**components** of $\mathbf{v}$, respectively. The position vectors $\mathbf{u} = \langle v_1, v_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ are equal if and only if $u_1 = v_1$ and $u_2 = v_2$.

Magnitude of a Vector

Given the points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the **magnitude**, or **length**, of $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1\rangle$, denoted $|\vec{PQ}|$, is the distance between P and Q:

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1}$$

The magnitude of the position vector $\mathbf{v} = \langle v_1, v_2\rangle$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$

Vector Operations

Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$.

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2\rangle \quad \text{Vector addition} \tag{2}$$

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2\rangle \quad \text{Vector subtraction} \tag{3}$$

$$c\mathbf{u} = \langle cu_1, cu_2\rangle \quad \text{Scalar multiplication} \tag{4}$$

Unit Vectors and Vectors of a Specified Length

A **unit vector** is any vector with length 1. Given a nonzero vector $\mathbf{v}, \pm\frac{\mathbf{v}}{|\mathbf{v}|}$ are unit vectors parallel to $\mathbf{v}$. For a scalar $c > 0$, the vectors $\pm\frac{c\mathbf{v}}{|v|}$ are vectors of length c parallel to $\mathbf{v}$.

Properties of Vector Operations

Suppose $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

$\mathbf{u} + a = \mathbf{v} + \mathbf{u}$	Commutative property of addition	(5)
$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property of addition	(6)
$\mathbf{v} + \mathbf{0} = \mathbf{v}$	Additive identity	(7)
$\mathbf{v} + (-\mathbf{v}) = 0$	Additive identity	(8)
$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property 1	(9)
$(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$	Distributive property 2	(10)
$0\mathbf{v} = \mathbf{0}$	Multiplication by zero scalar	(11)
$c\mathbf{0} = \mathbf{0}$	Multiplication by zero vector	(12)
$1\mathbf{v} = \mathbf{v}$	Multiplicative identity	(13)
$a(c\mathbf{v}) = (ac)\mathbf{v}$	Associative property of scalar multiplication	(14)

12.2 Vectors in Three Dimensions

Distance Formula in xyz-Space

The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{15}$$

Spheres and Balls

A **sphere** centered at (a, b, c) with radius r is the set of points satisfying the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2 \tag{16}$$

A ball centered at (a, b, c) with radius r is the set of points satisfying the inequality

$$(x - a)^2 + (y - b)^2 + (z - c)^2 \leq r^2 \tag{17}$$

Vector Operations in $\mathbb{R}$

Let c be a scalar, $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$.

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle \qquad \text{Vector addition} \tag{18}$$

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3\rangle \qquad \text{Vector subtraction} \tag{19}$$

$$c\mathbf{u} = \langle cu_1, cu_2, c_3\rangle \tag{20}$$

Magnitude of a Vector

The **magnitude** (or **length**) of the vector $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$ is the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{21}$$

12.3 Dot Product

Dot Product

Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in two or three dimensions, their **dot product** is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta \tag{22}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ with $0 \leq \theta \leq \pi$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$, and θ is undefined.

Orthogonal Vectors

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. The zero vector is orthogonal to all vectors. In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.

Dot Product

Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{23}$$

Properties of the Dot Product

Suppose $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and let c be a scalar.

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \qquad \text{Commutative property} \tag{24}$$

$$c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) \qquad \text{Associative property} \tag{25}$$

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \tag{26}$$

(Orthogonal) Projection of u onto v

The **orthogonal projection of u onto v** , denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$, where $\mathbf{v} \neq \mathbf{0}$, is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right) \tag{27}$$

The orthogonal projection may also be computed with the formulas

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \text{scal}_{\mathbf{v}}\mathbf{u}\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} \tag{28}$$

where the **scalar component of u in the direction of v** is

$$\text{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} \tag{29}$$

Work

Let a constant force **F** be applied to an object, producing a displacement **d**. If the angle between **F** and **d** is θ, then the **work** done by the force is

$$W = |\mathbf{F}||\mathbf{d}|\cos\theta = \mathbf{F}\cdot\mathbf{d} \tag{30}$$

12.4 Cross Product

Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta \tag{31}$$

where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$. The direction of $\mathbf{u} \times \mathbf{v}$ is given by the **right-hand rule**: When you put the vectors tail to tail and let the fingers of your right hand curl from $\mathbf{u}$ to $\mathbf{v}$ the direction of $\mathbf{u} \times \mathbf{v}$ is the direction of your thumb, orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. When $\mathbf{u} \times \mathbf{v} = 0$, the direction of $\mathbf{u} \times \mathbf{v}$ undefined.

Geometry of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbb{R}^3$.

1. The vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \times \mathbf{v} = 0$.
2. If $\mathbf{u}$ and $\mathbf{v}$ are two sides of a parallelogram, then the area of the parallelogram is

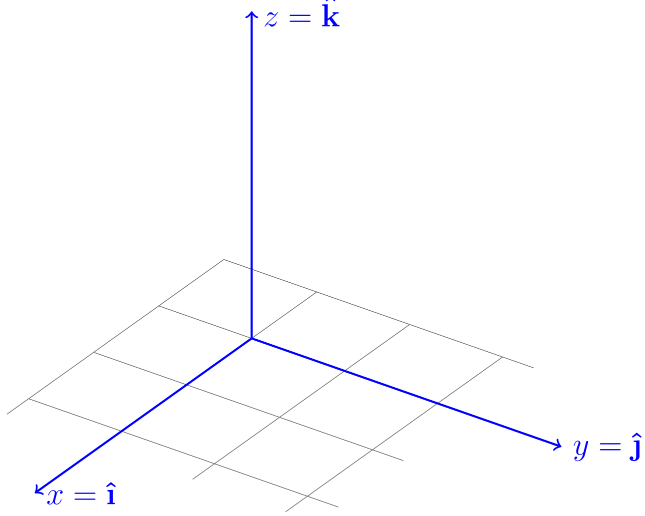
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta \tag{32}$$

Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^3$, and let a and b be scalars.

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= -(\mathbf{v} \times \mathbf{u}) && \text{Anticommutative property} && (33)\\
(a\mathbf{u}) \times (b\mathbf{v}) &= ab(\mathbf{u} \times \mathbf{v}) && \text{Associative property} && (34)\\
\mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) && \text{Distributive property} && (35)\\
(\mathbf{u} + \mathbf{v}) \times \mathbf{w} &= (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w}) && \text{Distributive property} && (36)
\end{aligned}$$

Cross Products of Coordinate Unit Vectors



$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = -(\hat{\mathbf{j}} \times \hat{\mathbf{i}}) = \hat{\mathbf{k}} \tag{37}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = -(\hat{\mathbf{k}} \times \hat{\mathbf{j}}) = \hat{\mathbf{i}} \tag{38}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = -(\hat{\mathbf{i}} \times \hat{\mathbf{k}}) = \hat{\mathbf{j}} \tag{39}$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0 \tag{40}$$

Evaluating the Cross Product

Let $\mathbf{u} = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}$ and $\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$. Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{\mathbf{k}} \tag{41}$$

12.5 Lines and Curves in Space

Equation of a Line

An equation of the line passing through the point $P_0(x_0,\, y_0,\, z_0)$ in the direction of the vector $\mathbf{v} = \langle a,\, b,\, c\rangle$ is $\mathbf{r} = r_0 + t\mathbf{v}$, or

$$\langle x,\, y,\, z\rangle = \langle x_0,\, y_0,\, z_0\rangle + t\langle a,\, b,\, c\rangle, \quad \text{for} \quad -\infty < t < \infty \tag{42}$$

Equivalently, the parametric equations of the line are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for} \quad -\infty < t < \infty \tag{43}$$

Limit of a Vector-Valued Function

A vector-valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written $\lim\limits_{t\to a} \mathbf{r}(t) = \mathbf{L}$, provided $\lim\limits_{t\to a} |\mathbf{r}(t) - \mathbf{L}| = 0$

12.6 Calculus of Vector-Valued Functions

Derivative and Tangent Vector

Let $\mathbf{r}(t) = f(t)\hat{\mathbf{\imath}} + g(t)\hat{\mathbf{\jmath}} + h(t)\hat{\mathbf{k}}$, where f, g, and h are differentiable functions on $(a\,b)$. Then $\mathbf{r}$ has a **derivative** (or is **differentiable**) on $(a\,b)$ and

$$\mathbf{r}'(t) = f'(i)\hat{\mathbf{\imath}} + g'(t)\hat{\mathbf{\jmath}} + h'(t)\hat{\mathbf{k}} \tag{44}$$

Provided $\mathbf{r}'(t) \neq \mathbf{0}, \mathbf{r}'(t)$ is a **tangent vector** (or velocity vector) at the point corresponding to $\mathbf{r}$.

Unit Tangent Vector

Let $\mathbf{r} = f(t)\hat{\mathbf{\imath}} + g(t)\hat{\mathbf{\jmath}} + h(t)\hat{\mathbf{k}}$ be a smooth parameterized curve, for $a \leq t \leq b$. The **unit tangent vector** for a particular value of t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \tag{45}$$

Derivative Rules

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector-valued functions and let f be a differentiable scalar-valued function, all at a point t. Let $\mathbf{c}$ be a constant vector. The following rules apply.

$$\frac{d}{dt}(\mathbf{c}) = \mathbf{0} \quad \text{Constant Rule} \tag{46}$$

$$\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t) \quad \text{Sum Rule} \tag{47}$$

$$\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t) \quad \text{Product Rule} \tag{48}$$

$$\frac{d}{dt}(\mathbf{u}(f(t))) = \mathbf{u}'(f(t))f'(t) \quad \text{Chain Rule} \tag{49}$$

$$\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \quad \text{Dot Product Rule} \tag{50}$$

$$\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) \quad \text{Cross Product Rule} \tag{51}$$

Indefinite Integral of a Vector-Valued Function

Let $\mathbf{r} = f\hat{\mathbf{i}} + g\hat{\mathbf{j}} + h\hat{\mathbf{k}}$ be a vector function and let $\mathbf{R} = F\hat{\mathbf{i}} + G\hat{\mathbf{j}} + H\hat{\mathbf{k}}$, where F, G, and H are antiderivatives of f, g, and h, respectively. The **indefinite integral** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C} \tag{52}$$

where $\mathbf{C}$ is an arbitrary constant vector.

Definite Integral of a Vector-Valued Function

Let $\mathbf{r}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$, where f, g, and h are integrable on the interval $[a, b]$.

$$\int \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int_a^b g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int_a^b h(t)\,dt\right]\hat{\mathbf{k}} \tag{53}$$

12.7 Motion In Space

Position, Velocity, Speed, Acceleration

Let the position of an object moving in three-dimensional space be given by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $t \geq 0$. The **velocity** of the object is

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle \tag{54}$$

The **speed** of the object is the scalar function

$$|\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \tag{55}$$

The **acceleration** of the object is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Motion with Constant $|\mathbf{r}|$

Let $\mathbf{r}$ describe a path on which $|\mathbf{r}|$ is constant (motion on a circle or sphere centered at the origin). Then, $\mathbf{r} \cdot \mathbf{v} = 0$, which means the position vector and the velocity vector are orthogonal at all times for which the functions are defined.

Two-Dimensional Motion in a Gravitational Field

Consider an object moving in a plane with horizontal x-axis and a vertical y-axis, subject only to the force of gravity. Given the initial velocity $\mathbf{v}(0) = \langle u_0, v_0 \rangle$ and the initial position $\mathbf{r}(0) = \langle x_0, y_0 \rangle$, the velocity of the object, for $t \geq 0$, is

$$\mathbf{v}(t) = \langle x'(t), y'(t) \rangle = \langle u_0, -gt + v_0 \rangle \tag{56}$$

and the position is

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \left\langle u_0\, t + x_0, -\frac{1}{2}gt^2 + v_0 t + y_0 \right\rangle \tag{57}$$

Two-Dimensional Motion

Assume an object traveling over horizontal ground, acted on only by the gravitational force, has an initial position $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ and initial velocity

$\langle u_0,\, v_0\rangle = \langle |\mathbf{v}_0| \cos\alpha,\, |\mathbf{v}_0| \sin\alpha\rangle$. The trajectory, which is a segment or a parabola, has the following properties.

$$\text{time of flight} = T = \frac{2|\mathbf{v}_0| \sin\alpha}{g} \tag{58}$$

$$\text{range} = \frac{|\mathbf{v}_0| \sin 2\alpha}{g} \tag{59}$$

$$\text{maximum height} = y\left(\frac{T}{2}\right) = \frac{(|\mathbf{v}_0| \sin\alpha)^2}{2g} \tag{60}$$

12.8 Length of Curves

Consider the parameterized curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f', g', and h' are continuous, and the curve is traversed once for $a \leq t \leq b$. The **arc length** of the curve between $(f(a), g(a), h(a))$ and $(f(b), g(b), h(b))$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt = \int_a^b |\mathbf{r}'(\mathbf{t})|\, dt \tag{61}$$

Arc Length of a Polar Curve

Let f have a continuous derivative on the interval $[\alpha, \beta]$. The **arc length** of the polar curve $r = f(\theta)$ on $[\alpha,\ \beta]$ is

$$L = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta. \tag{62}$$

Arc Length as a Function of a Parameter

Let $\mathbf{r}$(t) describe a smooth curve, for $t \geq a$. The arc length is given by

$$s(t) = \int_a^t |\mathbf{v}(u)|\, du, \tag{63}$$

where $|\mathbf{v}| = |\mathbf{r}'|$. Equivalently, $\frac{ds}{dt} = |\mathbf{v}t| > 0$. If $|\mathbf{v}(t)| = 1$, for all $t \geq a$, then the parameter t corresponds to arc length.

Arc Length as a Function of a Parameter

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12.9 Curvature and Normal Vectors

Curvature

Let $\mathbf{r}$ describe a smooth parameterized curve. If s denotes arc length and $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$ is the unit tangent vector, the **curvature** is $\kappa(s) = |\frac{d\mathbf{T}}{ds}|$

Curvature Formula

Let $\mathbf{r}(t)$ describes a smooth parameterized curve, where t is any parameter. If $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{T}$ is the unit tangent vector, then the curvature is

$$\kappa(t) = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \left|\frac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right| \tag{65}$$

Alternative Curvature Formula

Let $\mathbf{r}$ be the position of an object moving on a smooth curve. The **curvature** at a point on the curve is

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}, \tag{66}$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{a} = \mathbf{v}'$ is the acceleration.

Principal Unit Normal Vector

Let $\mathbf{r}$ describe a smooth parameterized curve. The **principal unit normal vector** at a point P on the curve at which $\kappa \neq 0$ is

$$\mathbf{N}(s) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} \tag{67}$$

In practice, we use the equivalent formula

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \tag{68}$$

evaluated at the value of t corresponding to P.

Properties of the Principal Unit Normal Vector

Let $\mathbf{r}$ describe a smooth parameterized curve with unit tangent vector $\mathbf{T}$ and principal unit normal vector $\mathbf{N}$.

1. $\mathbf{T}$ and $\mathbf{N}$ are orthogonal at all points of the curve; that is, $\mathbf{T}(t)\cdot\mathbf{N}(t) = 0$, at all points where $\mathbf{N}$ is defined.

2. The principal unit normal vector points to the inside of the curve—in the direction that the curve is turning.

Tangential and Normal Components of the Acceleration

The acceleration vector of an object moving in space along a smooth curve has the following representation in terms of its **tangential component** a_T (in the direction of $\mathbf{T}$) and its normal component a_N (in the direction of $\mathbf{N}$):

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}, \tag{69}$$

where $a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|}$ and $a_T = \frac{d^2s}{dt^2}$.

Unit Binormal Vector and Torsion

Let C be a smooth parameterized curve with unit tangent and principal unit normal vectors $\mathbf{T}$ and $\mathbf{N}$, respectively. Then, at each point of the curve at which the curvature is nonzero, the **unit binormal vector** is

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} \tag{70}$$

and the **torsion** is

$$\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N} \tag{71}$$

Formulas for Curves in Space

1. Position function: $\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$

2. Velocity: $\mathbf{v} = \mathbf{r}'$

3. Acceleration: $\mathbf{a} = \mathbf{v}'$

4. Unit tangent vector: $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

5. Principal unit normal vector: $\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$ (provided $d\mathbf{T}/dt \neq \mathbf{0}$)

6. Curvature: $\kappa = \frac{d\mathbf{T}}{ds} = \frac{1}{|\mathbf{v}|}|\frac{d\mathbf{T}}{dt}| = |\frac{\mathbf{T}'(t)}{\mathbf{r}'(t)}| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$

7. Components of acceleration: $\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}$, where $a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|}$ and $a_T = \frac{d^2s}{dt^2} = \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}|}$

8. Unit binormal vector: $\mathbf{B} = \mathbf{B} \times \mathbf{N} = \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|}$

9. Torsion $\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \frac{(\mathbf{v}\times\mathbf{a})\cdot\mathbf{a}'}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{(\mathbf{r}'\times\mathbf{r}'')^2}$

Chapter 13: Functions of Several Variables

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Contents

13.1 Planes and Surfaces

Plane in $\mathbb{R}^3$

Given a fixed point P_0 and a nonzero **normal vector n**, the set of points P in $\mathbb{R}^3$ for which $\vec{P_0 P}$ is orthogonal to **n** is called a **plane**.

General Equation of a Plane in $\mathbb{R}^3$

The plane passing through the point $P_0(x_0, y_0, z_0)$ with a nonzero normal vector $\mathbf{n} = \langle a, b, c\rangle$ is described by the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d \tag{1}$$

where $d = ax_0 + by_0 + cz_0$.

Parallel and Orthogonal Planes

Two distinct planes are **parallel** if their respective normal vectors are parallel (that is, the normal vectors are scalar multiples of each other). Two planes are **orthogonal** if their respective normal vectors are orthogonal (that is, the dot product of the normal vectors is zero).

Cylinder

Given a curve C in a plane P and a line ℓ not in P, a **cylinder** is the surface consisting of all lines parallel to ℓ that pass through C.

Trace

A **trace** of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The trace in the coordinate planes are called **xy-trace**, the **xz-trace**, and the **yz-trace**.

Quadratic Surfaces

Name	**Standard Equation**	**Features**
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	All traces are ellipses.
Elliptic paraboloid	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	Traces with $z = z_0 > 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are parabolas.
Hyperboloid of one sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Traces with $z = z_0$ are ellipses for all z_0. Traces with $x = x_0$ or $y = y_0$ are hyperbolas.
Hyperboloid of two sheets	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Traces with $z = z_0$ with $\lvert z_0 \rvert > \lvert c \rvert$ are ellipses. Traces with $x = x_0$ and $y = y_0$ are hyperbolas.
Elliptic cone	$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$	Traces with $z = z_0 \neq 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are hyperbolas or intersecting lines.
Hyperbolic paraboloid	$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$	Traces with $z = z_0 \neq 0$ are hyperbolas. Traces with $x = x_0$ or $y = y_0$ are parabolas.

13.2 Graphs and Level Curves

Function, Domain, and Range with Two Independent Variables

A **function** $z = f(x, y)$ assigns to each point (x, y) in a set D in $\mathbb{R}^2$ a unique real number z in a subset of $\mathbb{R}$. The set D is the **domain** of f. The **range** of f is the set of real numbers z that are assumed as the points (x, y) vary over the domain.

Function, Domain, and Range with n Independent Variables

The **function** $y = f(x_1, x_2, \ldots, x_n)$ assigns a unique real number y to each point $(x_1, x_2, \ldots, x_n)$ in a set D in $\mathbb{R}^n$. The set D is the **domain** of f. The **range** is the set of real numbers y that are assumed as the points $(x_1, x_2, \ldots, x_n)$ vary over the domain.

13.3 Limits and Continuity

Limit of a Function of Two Variables

The function f has the **limit** L as $P(x, y)$ approaches $P_0(a, b)$ written

$$\lim (x, y) \to (a, b) f(x, y) = \lim P \to P_0 f(x, y) = L \tag{2}$$

if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \tag{3}$$

whenever (x, y) is in the domain of f and

$$0 < |PP_0| = \sqrt{(x-a)^2 + (y-b)^2} < \delta \tag{4}$$

Limits of Constant and Linear Functions

Let a, b, and c be real numbers.

1. Constant functions $f(x, y) = c : \lim_{(x, y)\to(a, b)} c = c$
2. Linear function $f(x, y) = x : \lim_{(x, y)\to(a, b)} x = a$
3. Linear function $f(x, y) = y : \lim_{(x, y)\to(a, b)} y = b$

Limit Laws and Functions of Two Variables

Let L and M be real numbers and suppose that $\lim_{(x, y)\to(a, b)} f(x, y) = L$ and $\lim_{(x, y)\to(a, b)} g(x, y) = M$. Assume c is a constant, and $\forall m, n \in \mathbb{Z}$.

1 Sum $\lim_{(x, y)\to(a, b)} (f(x, y) + g(x, y)) = L + M$

2 Difference $\lim_{(x, y)\to(a, b)} (f(x, y) - g(x, y)) = L - M$

3 Constant multiple $\lim_{(x, y)\to(a, b)} cf(x, y) = cL$

4 Product $\lim_{(x, y)\to(a, b)} f(x, y) \cdot g(x, y) = L \cdot M$

5 Quotient $\lim_{(x, y)\to(a, b)} \left[\frac{f(x, y)}{g(x, y)}\right] = \frac{L}{M}$

6 Power $\lim_{(x,\,y)\to(a,\,b)} (f(x,\,y))^n = L^n$

7 $^m/_n$ **Power** If m and n have no common factors and $n \neq 0$, then $\lim_{(x,\,y)\to(a,\,b)} [f(x,\,y)]^{m/n} = L^{m/n}$, where we assume $L > 0$ if n is even.

Interior and Boundary Points

Let R be a region in $\mathbb{R}^2$. An **interior point** P of R lies entirely within R, which means it is possible to find a disk centered at P that contains only points of R.
A **boundary point** Q of R lies on the edge of R in the sense that *every* disk centered at Q contains at least one point in R and at least one point not in R.

Open and Closed Sets

A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points.

Two-Path Test for Nonexistence of Limits

If $f(x,\,y)$ approaches two different values as $(x,\,y)$ approaches $(a,\,b)$ along two different paths in the domain of f, then $\lim_{(x,\,y)\to(a,\,b)} f(x,\,y)$ does not exist.

Continuity

A function f is continuous at the point $(a,\,b)$ provided

1. f is defined at $(a,\,b)$.
2. $\lim_{(x,\,y)\to(a,\,b)} f(x,\,y)$ exists.
3. $\lim_{(x,\,y)\to(a,\,b)} f(x,\,y) = f(a,\,b)$.

Continuity of Composite Functions

If $u = g(x,\,y)$ is continuous at $(a,\,b)$ and $z = f(u)$ is continuous at $g(a,\,b)$, then the composite function $z = f(g(x,\,y))$ is continuous at $(a,\,b)$.

13.4 Partial Derivatives

The **partial derivative of** f **with respect to** x **at the point (a, b)** is

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}. \tag{5}$$

The **partial derivative of** f **with respect to** y **at the point (a, b)** is

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}. \tag{6}$$

provided these limits exists.

Equality of Mixed Partial Derivatives

Assume that f is defined on an open set D of $\mathbb{R}^2$, and f_{xy} and f_{yx} are continuous throughout D. Then $f_{xy} = f_{yx}$ at all points of D.

Differentiability

The function $z = f(x,\, y)$ is **differentiable at** $(a,\, b)$ provided $f_x(a,\, b)$ and $f_y(a,\, b)$ exist and the change $\Delta z = f(a + \Delta x,\, b + \Delta y) - f(a,\, b)$ equals

$$\Delta z = f_x(a,\, b)\Delta x + f_y(a,\, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y, \tag{7}$$

where for fixed a and b, ε_1 and ε_2 are functions that depend only on Δx and Δy, with $(\varepsilon_1, \varepsilon_2) \to (0,\, 0)$ as $(\Delta x,\, \Delta y) \to (0,\, 0)$. A function is **differentiable** on an open set R if it is differentiable at every point on R.

Conditions for Differentiability

Suppose the function f has partial derivatives f_x and f_y defined on an open set containing $(a,\, b)$, with f_x and f_y continuous $(a,\, b)$. Then f is differentiable at $(a,\, b)$.

Differentiability Implies Continuity

If a function f is differentiable at $(a,\, b)$, then it is continuous at $(a,\, b)$

13.5 The Chain Rule

Chain Rule (One Independent Variable)

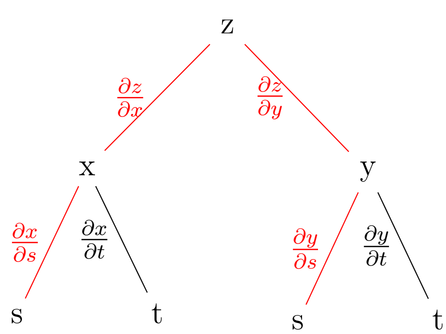
Let z be a differentiable function of x and y on its domain, where x and y are differentiable functions of t on an interval I. Then

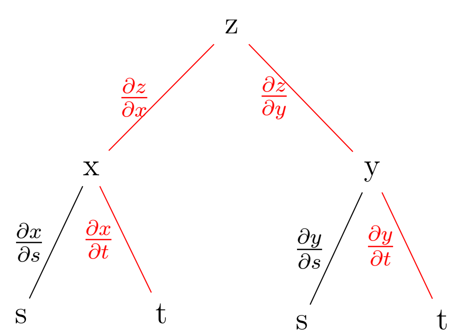
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{8}$$

Chain Rule (Two Independent Variables)

Let z be a differentiable function of x and y, where x and y are differentiable functions of s and t. Then

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \tag{9}$$





Implicit Differentiation

Let F be differentiable on its domain and suppose that $F(x, y) = 0$ defines y as a differentiable function of x. Provided $F_y \neq 0$.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \tag{10}$$

13.6 Directional Derivatives and the Gradient

Directional Derivative

Let f be a differentiable at $(a,\, b)$ and let $\mathbf{u} = \langle \cos\theta,\, \sin\theta \rangle$ be a unit vector in the xy-plane. The **directional derivatives of f at** $(a,\, b)$ in the direction of $\mathbf{u}$ is

$$D_u f(a,\, b) = \lim_{h \to 0} \frac{f(a + h\cos\theta,\, b + h\sin\theta) - f(a,\, b)}{h} \tag{11}$$

provided the limit exits.

Directional Derivative

Let f be differentiable on $(a,\, b)$ and let $\mathbf{u} = \langle u_1,\, u_2 \rangle$ be a unit vecor in the $x,\, y$-plane. The **directional derivative of f at a** $(a,\, b)$ **in the direction of** u is

$$D_u f(a,\, b) = \langle f_x(a,\, b),\, f_y(a,\, b) \rangle \cdot \langle u_1,\, u_2 \rangle \tag{12}$$

Gradient (Two Dimensions)

Let f be differentiable at the point $(x,\, y)$. The **gradient** of f at $(x,\, y)$ is the vector-valued function

$$\nabla f(x,\, y) = \langle f_x(x,\, y),\, f_y(x,\, y) \rangle = f_x(x,\, y)\hat{\mathbf{i}} + f_y(x,\, y)\hat{\mathbf{j}} \tag{13}$$

Directions of Change

Let f be differentiable at $(a,\, b)$ with $\nabla f(a,\, b) \neq \mathbf{0}$

1. f has its maximum rate of increase at $(a,\, b)$ in the direction of the gradient $\nabla f(a,\, b)$. The rate of increase in this direction is $|\nabla f(a,\, b)|$
2. f has its maximum rate of decrease at $(a,\, b)$ in the direction of $-\nabla f(a,\, b)$. The rate of decrease in this direction is $-|\nabla f(a,\, b)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a,\, b)$.

The Gradient and Level Curves

Given a function f differentiable at $(a,\, b)$, the line tangent to the level curve of f at $(a,\, b)$ is orthogonal to the gradient $\nabla f(a,\, b)$, provided $\nabla f(a,\, b) \neq \mathbf{0}$.

Gradient and Directional Derivative in Three Dimensions

Let f be differentiable at the point $(x,\, y,\, z)$. The **gradient** of f at $(x,\, y,\, z)$ is the vector-valued function

$$\begin{aligned} \nabla f(x,\, y,\, z) &= \langle f_x(x,\, y,\, z),\, f_y(x,\, y,\, z),\, f_z(x,\, y,\, z)\rangle &\quad (14) \\ &= f_x(x,\, y,\, z)\hat{\mathbf{i}} + f_y(x,\, y,\, z)\hat{\mathbf{j}} + f_z(x,\, y,\, z)\hat{\mathbf{k}} &\quad (15) \end{aligned}$$

The **directional derivative** of f in the direction of the unit vector $\mathbf{u} = \langle u_1,\, u_2,\, u_3\rangle$ at the point $(a,\, b,\, c)$ is $D_u f(a,\, b,\, c) = \nabla f(a,\, b,\, c) \cdot \mathbf{u}$

13.7 Tangent Planes and Linear Approximation

Let F be differentiable at the point $P_0(a, b, c)$ with $\nabla F(a, b, c) \neq \mathbf{0}$. The plane tangent to the surface $F(x, y, z) = 0$ at P_0, called the **tangent plane**, is the plane passing through P_0 orthogonal $\nabla F(a, b, c)$. An equation of the tangent plane is

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0. \tag{16}$$

Tangent Plane for $z = f(x, y)$

Let f be differentiable at the point (a, b). An equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b) \tag{17}$$

Linear Approximation

Let f be differentiable at (a, b). The linear approximation to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is the tangent plane at that point, given by the equation

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b) \tag{18}$$

The Differential dz

Let f be differentiable at the point (a, b). The change in $z = f(x, y)$ as the independent variables change from (a, b) to $(a + dx, b + dy)$ is denoted by the differential dz:

$$\Delta z \approx dz = f_x(a, b)dx + f_y(a, b)dy \tag{19}$$

13.8 Maximum/Minimum Problems

Local Maximum/Minimum Values

A function f has a **local maximum value** at (a, b) if $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f in some open disk centered at (a, b). A function f has a **local minimum value** at (a, b) if $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f in some open disk centered at (a, b). Local maximum and local minimum values are also called **local extreme values** or **local extrema**.

Derivatives and Local Maximum/Minimum Values

If f has a local maximum or minimum value at (a, b) and the partial derivatives f_x and f_y exist at (a, b) then $f_x(a, b) = f_y(a, b) = 0$.

Critical Point

An interior point $(a,\ b)$ in the domain of f is a **critical point** of f is either

1. $f_x(a,\ b) = f_y(a,\ b) = 0$, or
2. one (or both) of f_x or f_y does not exist at $(a,\ b)$

Saddle Point

A function f has a **saddle point** at a critical point $(a,\ b)$ if, in every open disk centered at $(a,\ b)$, there are points $(x,\ y)$ for which $f(x,\ y) > f(a,\ b)$ and points for which $f(x,\ y) < f(a,\ b)$

Second Derivative Test

Suppose that the second partial derivative of f are continuous throughout an open disk centered at the point $(a,\ b)$, where $f_x(a,\ b) = f_y(a,\ b) = 0$. Let $D(x,\ y) = f_{xx}(x,\ y)f_{yy}(x,\ y) - (f_{xy}(x,\ y))^2$.

1. If $D(a,\ b) > 0$ and $f_{xx}(a,\ b) < 0$, then f has a local maximum value at $(a,\ b)$.
2. If $D(a,\ b) > 0$ and $f_{xx}(a,\ b) > 0$, then f has a local minimum value at $(a,\ b)$.

3. If $D(a, b) < 0$, then f has a saddle point at (a, b).

4. If $D(a, b) = 0$, then the test is inconclusive.

Absolute Maximum/Minimum Values

If $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f, then f has an **absolute maximum value** at (a, b). If $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f, then f has an **absolute minimum value** at (a, b).

Finding Absolute Maximum/Minimum Values on Closed, Bounded Sets

Let f be continuous on a closed bounded set R in $\mathbb{R}^2$. To find the absolute maximum and minimum values of f on R:

1. Determine the values of f at all critical points in R.

2. Find the maximum and minimum values of f on the boundary of R.

3. The greatest function values found in Step 1 and 2 is the absolute maximum value of f on R, and the least function value found in Steps 1 and 2 is the absolute minimum value of f on R.

13.9 Lagrange Multipliers

Parallel Gradients (Ball Park Theorem)

Let f be a differentiable function in a region of $\mathbb{R}^2$ that contains the smooth curve C given by $g(x, y) = 0$. Assume that f has a local extreme value (relative to values of f on C) at a point $P(a, b)$ on C. Then $\nabla f(a, b)$ is orthogonal to the line tangent to C at P. Assuming $\nabla g(a, b) \neq 0$, it follows that there is a real number λ (called a **Lagrange multiplier**) such that $\nabla f(a, b) = \lambda \nabla g(a, b)$.

Method of Lagrange Multipliers in Two Variables

Let the objective function f and the constraint function g be differentiable on a region of $\mathbb{R}^2$ with $\lambda g(x, y) \neq 0$ on the curve $g(x, y) = 0$. To locate the maximum and minimum values of f subject to the constraint $g(x, y) = 0$, carry out the following steps.

1. Find the values of x, y and λ (if they exist) that satisfy the equations

$$\nabla f(x,\, y) = \lambda \nabla g(x,\, y) \quad \text{and} \quad g(x,\, y) = 0 \tag{20}$$

2. Among the values $(x,\, y)$ found in Step 1, select the largest and smallest corresponding function values, which are the maximum and minimum values of f subject to the constraint.

Method of Lagrange Multipliers in Three Variables

Let f and g be differentiable on a region of $\mathbb{R}^3$ with $\nabla g(x,\, y,\, z) neq 0$ on the surface $g(x,\, y,\, z) = 0$. To locate the maximum and minimum values of f subject to the constraint $g(x,\, y,\, z) = 0$, carry out the following steps.

1. Find the values of $x,\, y,\, z$ and λ that satisfy the equations

$$\nabla f(x,\, y,\, z) = \lambda \nabla g(x,\, y,\, z) \quad \text{and} \quad g(x,\, y,\, z) = 0 \tag{21}$$

2. Among the points $(x,\, y,\, z)$ found in Step 1, select the largest and smallest corresponding values of the objective function. These values are the maximum and minimum values of f subject to the constraint.

Chapter 14: Multiple Integration

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14.1 Double Integrals over Rectangular Regions

Volumes and Double Integrals

A function f defined on a rectangular region R in the xy-plane is **integratabtle** on R if $\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$ exists fora ll partitions of R and for all choices of (x_k^*, y_k^*) within those partitions. The limit is the **double integral of f over R**, which we write

$$\iint\limits_R f(x, y)\, dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \tag{1}$$

If f is nonnegative on R, then the double integral equals the volume of the solid bounded by $z = f(x, y)$ and the xy-plane over R.

Double Integrals on Rectangular Regions

Let f be continuous on the rectangular region $R = \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\}$. The double integral of f over R may be evaluated by either of two iterated integrals:

$$\iint\limits_R f(x, y)\, dA = \int_c^d \int_a^b dx\, dy = \int_a^b \int_c^d dy\, dx \tag{2}$$

Average Value of a Function over a Plane Region

The **average value** of an integrable function f over a region R is

$$\bar{f} = \frac{1}{\text{area of } R} \iint\limits_R f(x, y)\, dA \tag{3}$$

14.2 Double Integrals over General Regions

Let R be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines $x = a$ and $x = b$. If f is continuous on R, then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx \tag{4}$$

Let R be a region bounded on the left and right by the graphs of the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and the lines $y = c$ and $y = d$. If f is continuous on R, then

$$\iint\limits_R f(x,\, y)\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x,\, y)\, dx\, dy \tag{5}$$

Areas of Regions by Double Integrals

Let R be a region in the xy-plane. Then

$$\text{area of } R = \iint\limits_R \tag{6}$$

14.3 Double Integrals in Polar Coordinates

Double Integrals over Polar Rectangular Region

Let f be continuous on the region in the xy-plane $R = \{(r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$, where $\beta - \alpha \leq 2\pi$. Then

$$\iint_R f(r, \theta)\, dA = \int_\alpha^\beta \int_a^b f(r, \theta)\ r\, dr\, d\theta \tag{7}$$

Double Integrals over More General Polar Regions

Let f be continuous on the region in the xy-plane

$$R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\} \tag{8}$$

where $0 < \beta - \alpha \leq 2\pi$. Then

$$\iint_R f(r, \theta)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r, \theta) r\, dr\, d\theta. \tag{9}$$

Area of Polar Regions

The area of the region $R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\}$, where $0 < \beta - \alpha \leq 2\pi$, is

$$A = \iint_R dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r\, dr\, d\theta \tag{10}$$

14.4 Triple Integrals

Let f be continuous over the region

$$D = \{(x,\, y,\, z) : a \leq x \leq b, g(x) \leq y \leq h(x),\, G(x,\, y) \leq z \leq H(x,\, y)\} \quad (11)$$

where g, h, G, and H are continuous functions. Then f is integrable over D and the triple integral is evaluated as the iterated integral

$$\iiint\limits_D f(x,\, y,\, z)\, dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x,\, y,\, z)\, dz\, dy\, dx. \quad (12)$$

Average Value of a Function of Three Variables

If f is continuous on a region D of $\mathbb{R}^3$, then the average value of over D is

$$\bar{f} = \frac{1}{\text{volume } (D)} \iiint\limits_D f(x,\, y,\, z)\, dV \quad (13)$$

14.5 Triple Integrals in Cylindrical and Spherical Coordinates

Transformations Between Cylindrical and Rectangular Coordinates

Rectangular → Cylindrical	Cylindrical → Rectangular
$r^2 = x^2 + y^2$	$x = r\cos\theta$
$\tan\theta = {}^{y}\!/_{x}$	$y = r\sin\theta$
$z = z$	$z = z$

Triple Integrals in Cylindrical Coordinates

Let f be continuous over the region

$$D = \{(r,\,\theta,\,z) : g(\theta) \le r \le h(\theta),\, \alpha \le \theta \le \beta,\, G(x,\,y) \le z \le H(x,\,y).\} \quad (14)$$

Then f is integrable over D and the triple integral of f over D in cylindrical coordinates is

$$\iiint_D f(r,\,\theta,\,z)\,dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta,\,r\sin\theta)}^{H(r\cos\theta,\,r\sin\theta)} f(r,\,\theta,\,z)\,dz\ r\,dr\,d\theta \quad (15)$$

Tranformations Between Spherical and Rectangular Coordinates

Rectangular → Spherical	Spherical → Rectangular
$\rho^2 = x^2 + y^2 + z^2$	$x = \rho\sin\varphi\cos\theta$
Use trigonometry to find φ and θ	$y = \rho\sin\varphi\sin\theta$
	$z = \rho\cos\varphi$

Triple Integrals in Spherical Coordinates

Let f be continuous over the region

$$D = \{(\rho,\,\varphi,\,\theta) : g(\varphi,\,\theta) \le \rho \le h(\varphi,\,\theta),\, a \le \varphi b,\, \alpha \le \theta \le \beta\} \quad (16)$$

Then f is integrable over D, and the triple integral of f over D in spherical coordinates is

$$\iiint\limits_D f(\rho,\, \varphi,\, \theta) dV = \int_\alpha^\beta \int_a^b \int_{g(\varphi,\,\theta)}^{h(\varphi,\,\theta)} f(\rho,\, \varphi,\, \theta) \rho^2 \sin\varphi \; d\rho \, d\varphi \, d\theta \qquad (17)$$

14.6 Integrals for Mass Calculations

Center of Mass in One Dimension

Let ρ be an integrable density function on the interval $[a, b]$ (which represents a thin rod or wire). The **center of mass** is location at the point $\bar{x} = \frac{M}{m}$, where the **total moment** M and mass m are

$$M = \int_a^b x\rho(x)\,dx \quad \text{and} \quad m = \int_a^b \rho(x)\,dx \tag{18}$$

Center of Mass in Two Dimensions

Let ρ be integrable density function defined over a closed bounded region R in $\mathbb{R}^2$. The coordinates of the center of mass of the object represented by R are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_R x\rho(x,\, y)\,dA \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m} = \iint_R y\rho(x,\, y)\,dA \tag{19}$$

Center of Mass in Three Dimensions

Let ρ be integrable density function on a closed bounded region D in $\mathbb{R}^3$. The coordinates of the center of mass of the region are

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m}\iiint_R x\rho(x,\, y,\, z)\,dV \tag{20}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1}{m}\iiint_R y\rho(x,\, y,\, z)\,dV \tag{21}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\iiint_R z\rho(x,\, y,\, z)\,dV \tag{22}$$

where $m = \iiint_D \rho(x,\, y,\, z)\,dV$ is the mass, and M_{yz}, M_{xz} and M_{xy} are the moments with respect to the coordinate planes.

Chapter 15: Vector Calculus

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15.1 Vector Fields

Vector Fields in Two Dimensions

Let f and g be defined on a region R of $\mathbb{R}^2$. A **vector field** in $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point in R a vector $\langle f(x,y),\, g(x,y)\rangle$. The vector field is written as

$$\mathbf{F}(x,y) = \langle f(x,y),\, g(x,y)\rangle \quad \text{or} \quad \mathbf{F}(x,y) = f(x,y)\hat{\mathbf{i}} + g(x,y)\hat{\mathbf{j}} \tag{1}$$

A vector field $\mathbf{F} = \langle f, g\rangle$ is continuous or differentiable on a region R of $\mathbb{R}^2$ if f and g are continuous or differentiable on R, respectively.

Radial Vector Fields in $\mathbb{R}^2$

Let $\mathbf{r} = \langle x,\, y\rangle$. A vector field of the form $\mathbf{F} = f(x,\, y)\mathbf{r}$, where f is a scalar-valued function, is a **radial vector field**. Of specific interest are the radial vector field

$$\mathbf{F}(x,\, y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x,\, y\rangle}{|\mathbf{r}|^p} \tag{2}$$

where p is a real number. At every point (except the origin), the vectors of this field are directed outward from the origin with the magnitude of $|\mathbf{F}| = \frac{1}{|\mathbf{r}|^{p-1}}$.

Vector Fields in Three Dimensions

Let $f,\ g,$ and h be defined on a region D of $\mathbb{R}^3$. A **vector field** in $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point in D a vector $\langle f(x,\, y,\, z),\, g(x,\, y,\, z),\, h(x,\, y,\, z)\rangle$. The vector field is written as

$$\mathbf{F}(x,\, y,\, z) = \langle f(x,\, y,\, z),\, g(x,\, y,\, z),\, h(x,\, y,\, z)\rangle \quad \text{or} \tag{3}$$

$$\mathbf{F}(x,\, y,\, z) = f(x,\, y,\, z)\hat{\mathbf{i}} + g(x,\, y,\, z)\hat{\mathbf{j}} + h(x,\, y,\, z)\hat{\mathbf{k}} \tag{4}$$

A vector field $\mathbf{F} = \langle f,\, g,\, h\rangle$ is continuous or differentiable on a region D of $\mathbb{R}^3$ if $f,\ g,\ h$ are continuous or differentiable on R, respectively. Of particular importance are the **radial vector fields**

$$\mathbf{F}(x,\, y,\, z) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x,\, y,\, z\rangle}{|\mathbf{r}|^p} \tag{5}$$

where p is a real number.

Gradient Fields and Potential Functions

Let $z = \varphi(x,\, y)$ and $w = \varphi(x,\, y,\, z)$ be differentiable functions on regions of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. The vector field $\mathbf{F} = \nabla\varphi$ is **gradient field**, and the function φ is a **potential function** for $\mathbf{F}$.

15.2 Line Integrals

Scalar Line Integral in the Plane, Arc Length Parameter

Suppose the scalar-valued function f is defined on the smooth curve C : $\mathbf{r}(s) = \langle x(s),\, y(s)\rangle$, parameterized by the arc length s. The **line integral of f over C** is

$$\int\limits_C f(x(s),\, y(s))\, ds = \lim_{\Delta\to 0} \sum_{k=1}^{n} f(x(s_k^*),\, y(s_k^*))\Delta s_k, \tag{6}$$

provided this limit exists over all partitions of C. When the limit exists, f is said to be **integrable** on C.

Evaluating Scalar Line Integrals in $\mathbb{R}^2$

Let f be continuous on a region containing a smooth curve C : $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$. Then

$$\int\limits_C f\, ds = \int_a^b f(x(t),\, y(t))|\mathbf{r}'(t)|\, dt \tag{7}$$

$$= \int_a^b f(x(t),\, y(t))\sqrt{x'(t)^2 + y'(t)^2}\, dt \tag{8}$$

Evaluating the Line Integral $\int\limits_C f\, ds$

1. Find a parametric description of C in the form $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$
2. Computer $|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$
3. Make substitutions for x and y in the integrand and evaluate an ordinary integral

$$\int\limits_C f\, ds = \int_a^b f(x(t),\, y(t))|\mathbf{r}'(t)|\, dt \tag{9}$$

Evaluating Scalar Line Integrals in $\mathbb{R}^3$

Let f be continuous on a region containing a smooth curve $C : \mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$, for $a \le t \le b$. Then

$$\int f\,ds = \int_a^b f(x(t),\, y(t),\, z(t))|\mathbf{r}'(t)|\,dt \tag{10}$$

$$= \int_a^b f(x(t),\, y(t),\, z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt \tag{11}$$

Line Integral of a Vector Field

Let $\mathbf{F}$ be a vector field that is continuous on a region containing a smooth oriented curve C parameterized by arc length. Let $\mathbf{T}$ be the unit tangent vector at each point of C consistent with the orientation. The line integral of $\mathbf{F}$ over C is $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$.

Different Forms of Line Integrals of Vector Fields

The line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ may be expressed in the following forms, where $\mathbf{F} = \langle f,\, g,\, h\rangle$, for $a \le t \le b$:

$$\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_a^b (f\,x'(t),\, g\,y'(t),\, h\,z'(t))\,dt \tag{12}$$

$$= \int_C f\,dx + g\,dy + h\,dz \tag{13}$$

$$= \int_C \mathbf{F}\cdot d\mathbf{r} \tag{14}$$

For line integrals in the plane, we let $\mathbf{F} = \langle f,\, g\rangle$ and assume C is parameterized in the form $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$. Then

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b (f\,x'(t) + g\,y'(t))\,dt = \int_C f\,dx + g\,dy = \int_C \mathbf{F}\cdot d\mathbf{r} \tag{15}$$

Work Done in a Force Field

Let $\mathbf{F}$ be a continuous force field in a region D of $\mathbb{R}^3$ and let $C : \mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$, for $a \le t \le b$, be a smooth curve in D with a unit tangent vector $\mathbf{T}$ consistent with the orientation. The work done in moving an object C in the positive direction is

$$W = \int\limits_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(\mathbf{t})\, dt \tag{16}$$

Circulation

Let $\mathbf{F}$ be a continuous vector field on a region D of $\mathbb{R}^3$ and let C be a closed smooth oriented curve in D. The **circulation** of $\mathbf{F}$ on C is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{T}$ is the unit vector tangent to C consistent with the orientation.

Flux

Let $F = \langle f, g\rangle$ be continuous vector field on a region R of $\mathbb{R}^2$. Let $C : \mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$, be a smooth oriented curve in R that does not intersect itself. The **flux** of the vector field across C is

$$\int\limits_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b (f\, y'(t) - g\, x'(t))\, dt, \tag{17}$$

where $n = \mathbf{T} \times \hat{\mathbf{k}}$ is the unit normal vector and $\mathbf{T}$ is the unit tangent vector consistent with the orientation. If C is a closed curve with counterclockwise orientation, $\mathbf{n}$ is the outward normal vector and the flux integral gives the **outward flux** across C.

15.3 Conservative Vector Fields

Simple and Closed Curves

Suppose a curve C (in $\mathbb{R}^2$ and $\mathbb{R}^3$) is described parametrically by $\mathbf{r}(t)$, where $a \le t \le b$. Then C is a **simple curve** if $\mathbf{r}(t_1) \ne \mathbf{r}(t_2)$ for all t_1 and t_2, with $a < t_1 < t_2 < b$; that is, C never intersects itself between its endpoints. The curve C is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$; that is, the initial and terminal points of C are the same.

Connected and Simply Connected Regions

An open region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is **connected** if it possible to connect any two points of R by a continuous curve lying in R. An open region R is **simply connected** if every closed simple curve in R can be deformed and contracted to a point in R.

Conservative Vector Field

A vector field F is said to be **conservative** on a region (in $\mathbb{R}^2$ or $\mathbb{R}^3$) if there exists a scalar function φ such that $\mathbf{F} = \nabla\varphi$ on that region.

Test for Conservative Vector Fields

Let $\mathbf{F} = \langle f,\, g,\, h\rangle$ be a vector field defined on a connected and simply connected region D of $\mathbb{R}^3$, where f, g, and h have continuous first partial derivatives on D. Then $\mathbf{F}$ is a conservative vector field on D (there is a potential function φ such that $\mathbf{F} = \nabla\varphi$) if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \text{and} \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y} \tag{18}$$

For vector fields in $\mathbb{R}^2$, we have the single condition $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

Finding Potential Functions in $\mathbb{R}^3$

Suppose $\mathbf{F} = \langle f,\, g,\, h\rangle$ is a conservative vector field. To find φ such that $\mathbf{F} = \nabla\varphi$, take the following steps:

1. Integral $\varphi_x = f$ with respect to x to obtain φ, which includes an arbitrary function $c(y,\, z)$.

2. Compute φ_y and equate it to g to obtain an expression for $c_y(y, z)$.
3. Integrate $c_y(y, z)$ with respect to y to obtain $c(y, z)$, including an arbitrary function $d(z)$.
4. Compute φ_z and equate it to h to get $d(z)$.

Beginning the procedure with $\varphi_y = g$ or $\varphi_z = h$ maybe be easier in some cases.

Fundamental Theorem for Line Integrals

Let $\mathbf{F}$ be a continuous vector field on an open connected region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$). There exists a potential function φ with $\mathbf{F} = \nabla\varphi$ (which means that $\mathbf{F}$ is conservative) if and only if

$$\int\limits_C \mathbf{F} \cdot \mathbf{T}\, ds = \int\limits_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A) \tag{19}$$

for all points A and B in R and all smooth oriented curves C from A to B.

Line Integrals on Closed Curves

Let R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) be an open region. Then $\mathbf{F}$ is a conservative vector field on R if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed smooth oriented curves C in R.

15.4 Green's Theorem

Let C be a simple closed smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA. \tag{20}$$

Two-Dimensional Curl

The **two-dimensional curl** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$. If the curl is zero throughout a region, the vector field is said to be **irrotational** on that region.

Area of a Plane Region by Line Integrals

Under the conditions of Green's Theorem, the area of a region R enclosed by a curve C is

$$\oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2} \oint_C (x\,dy - y\,dx) \tag{21}$$

Green's Thoerem, Flux Form

Let C be a simple closed smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\oint_C \mathbf{F} \cdot \mathbf{n}\,ds = \oint_C f\,dy - g\,dx = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) \tag{22}$$

where $\mathbf{n}$ is the outward unit normal vector on the curve.

Two-Dimensional Divergence

The **two-dimensional divergence** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$. If the divergence is zero thoughout a region, the vector field is said to be **source free** on that region.

15.5 Divergence and Curl

Divergence of a Vector Field

The **divergence** of a vector field $\mathbf{F} = \langle f,\, g,\, h\rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \tag{23}$$

If $\nabla\cdot\mathbf{F} = 0$, the vector field is **source free.**

Divergence of Radial Vector Fields

For a real number p, the divergence of the radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|r|^p} = \frac{\langle x,\, y,\, z\rangle}{(x^2+y^2+z^2)^{\frac{p}{2}}} \text{ is } \nabla\cdot\mathbf{F} = \frac{3-p}{|\mathbf{r}|^p} \tag{24}$$

Curl of a Vector Field

The **curl** of a vector field $\mathbf{F} = \langle f,\, g,\, h\rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\nabla\times\mathbf{F} = \operatorname{curl}\mathbf{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{\mathbf{k}} \tag{25}$$

If $\nabla\times\mathbf{F} = \mathbf{0}$, the vector field is **irrotational**.

Curl of a Conservative Vector Field

The **general rotation vector field** is $\mathbf{F} = \mathbf{a}\times\mathbf{r}$, where the nonzero constant vector $\mathbf{a} = \langle a_1,\, a_2,\, a_3\rangle$ is the axis of ration and $\mathbf{r} = \langle x,\, y,\, z\rangle$. For all nonzero choices of $\mathbf{a}$, $|\nabla\times\mathbf{F}| = 2|\mathbf{a}|$ and $\nabla\cdot\mathbf{F} = 0$. The constant angular speed of the vector field is

$$\omega = |\mathbf{a}| = \frac{1}{2}|\nabla\times\mathbf{F}| \tag{26}$$

Curl of a Conservative Vector Field

Suppose that $\mathbf{F}$ is a conservative vector field on an open region D of $\mathbb{R}^3$. Let $\mathbf{F} = \nabla\varphi$, where φ is a potential function with continuous second partial derivatives on D. Then $\nabla \times \mathbf{F} = \nabla \times \nabla\varphi = \mathbf{0}$; that is, the curl of the gradient is the zero vector and $\mathbf{F}$ is irrotational.

Divergence of the Curl

Suppose that $\mathbf{F} = \langle f,\, g,\, h\rangle$, where $f,\ g,$ and h have continuous second partial derivatives. Then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$: The divergence of the curl is zero.

Product Rule for the Divergence

Let u be a scalar-valued functino that is differentiable on a region D and let $\mathbf{F}$ be a vector field that is differentiable on D. Then

$$\nabla \cdot (u\mathbf{F}) = \nabla u \cdot \mathbf{F} + u(\nabla \cdot \mathbf{F}) \tag{27}$$

Properties of a Conservative Vector Field

Let $\mathbf{F}$ be a conservative vector field whose components have continuous second partial derivatives on an open connected region D in $\mathbb{R}^3$. Then $\mathbf{F}$ has the following equivalent properties.

1. There exists a potential function φ such that $\mathbf{F} = \nabla\varphi$
2. $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points A and B in D and all smooth oriented curves C from A and B.
3. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple smooth closed oriented curves C in D.
4. $\nabla \times \mathbf{F} = \mathbf{0}$ at all points of D.

15.6 Surface Integrals

Surface Integrals of Scalar-Valued Functions on Parameterized Surface

Let f be a continuous function on a smooth surface S given parametrically by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, where $R = \{(u, v) : a \le u \le b, c \le v \le d\}$. Assume also that the tangent vectors $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\rangle$, and $\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\rangle$ are continuous on R and the normal vectors $\mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R. Then the **surface integral** of the scalar-valued function f over S is

$$\iint_S f(x, y, z)\, dS = \iint_R f(x(u, v), y(u, v), z(u, v))|\mathbf{t}_u \times \mathbf{t}_v|\, dA \qquad (28)$$

If $f(x, y, z) = 1$, the integral equals the surface area of S.

Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces

Let f be a continuous function on a smooth surfaces S given by $z = g(x, y)$, for (x, y) in a region R. The surface integral of f over S is

$$\iint_S f(x, y, z)\, dS = \iint_R f(x, y, g(x, y))\sqrt{z_x^2 + z_y^2 + 1}\, dA \qquad (29)$$

If $f(x, y, z) = 1$, the surface integral equals the area of the surface.

Surface Integral of a Vector Field

Suppose $\mathbf{F} = \langle f, g, h\rangle$ is a continuous vector field on a region of $\mathbb{R}^3$ containing a smooth oriented surface S. If S is defined parametrically as $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, for (u, v) is a region R, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v)\, dA \qquad (30)$$

where $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\rangle$ and $\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\rangle$ are continuous on R, the normal vector $\mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R, and the direction of $\mathbf{n}$ is consistent with the orientation of S. If S is defined in the form $z = g(z,\, y)$, for $(x,\, y)$ in a region R, then

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_R (-f z_x - g z_y + h)\, dA \tag{31}$$

15.7 Stokes' Theorem

Let S be a smooth oriented surface in $\mathbb{R}^2$ with a smooth closed boundary C whose orientation is consistent with that of S. Assume that $\mathbf{F} = \langle f,\, g,\, h \rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS \tag{32}$$

where $\mathbf{n}$ is the unit vector normal to S determined by the orientation of S.

Curl $\mathbf{F} = \mathbf{0}$ Implies $\mathbf{F}$ is Conservative

Suppose that $\nabla \times \mathbf{F} = \mathbf{0}$ throughout an open simply connected region D of $\mathbb{R}^3$. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed simple smooth curves C in D and $\mathbf{F}$ is a conservative vector field on D.

15.8 Divergence Theorem

Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D in $\mathbb{R}^3$ enclosed by a smooth oriented surface S. Then

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint\limits_D \nabla \cdot \mathbf{F}\, dV \tag{33}$$

where $\mathbf{n}$ is the unit outward normal vector on S.

Divergence Theorem for Hollow Regions

Suppose the vector field $\mathbf{F}$ satisfies the conditions of the Divergence Theorem on a region D bounded by two smooth oriented surfaces S_1 and S_2, where S_1 lies within S_2. Let S be the entire boundary of $D(S = S_1 \cup S_2)$ and let $\mathbf{n_1}$ and $\mathbf{n}_2$ be the outward unit normal vectors for S_1 and S_2, respectively. Then

$$\iiint\limits_D \nabla \cdot \mathbf{F}\, dV = \iint \mathbf{F} \cdot \mathbf{n} dS = \iint\limits_{S_2} \mathbf{F} \cdot \mathbf{n_2}\, dS = \iint\limits_{S_1} \mathbf{F} \cdot \mathbf{n_1}\, dS \tag{34}$$

Calculus III: Multivariable Calculus
Complete Notes

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Contents

12 Vectors and Vector-Valued Functions

12.1 Vectors in the Plane

Vectors, Equal Vectors, Scalars, Zero Vector

Vectors are quantities that have both **length** (or **magnitude**) and **direction**. Two vectors are **equal** if they have the same magnitude and direction. Quantities having magnitude but no direction are called **scalars**. One exception is the **zero** vector, denoted **0**: It has length 0 and no direction.

Scalar Multiples and Parallel Vectors

Given a scalar c and a vector $\mathbf{u}$, the scalar multiple $c\mathbf{v}$ is a vector whose magnitude is $|c|$ multiplied by the magnitude of $\mathbf{v}$. If $c > 0$, then $c\mathbf{v}$ has the same direction as $\mathbf{v}$. If $c < 0$, then $c\mathbf{v}$ and $\mathbf{v}$ point in opposite directions. Two vectors are **parallel** if they are scalar multiples of each other.

Position Vectors and Vector Components

A vector $\mathbf{v}$ with its tail at the origin and head at the point (v_1, v_2) is called a **position vector** (or is said to be in **standard position**) and is written $\langle v_1, v_2 \rangle$. The real numbers v_1 and v_2 are the x- and y-**components** of $\mathbf{v}$, respectively. The position vectors $\mathbf{u} = \langle v_1, v_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are equal if and only if $u_1 = v_1$ and $u_2 = v_2$.

Magnitude of a Vector

Given the points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the **magnitude**, or **length**, of $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$, denoted $|\vec{PQ}|$, is the distance between P and Q:

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1}$$

The magnitude of the position vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$

Vector Operations

Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$.

$$\mathbf{u}+\mathbf{v} = \langle u_1+v_1, u_2+v_2\rangle \qquad \text{Vector addition} \tag{2}$$
$$\mathbf{u}-\mathbf{v} = \langle u_1-v_1, u_2-v_2\rangle \qquad \text{Vector subtraction} \tag{3}$$
$$c\mathbf{u} = \langle cu_1, cu_2\rangle \qquad \text{Scalar multiplication} \tag{4}$$

Unit Vectors and Vectors of a Specified Length

A **unit vector** is any vector with length 1. Given a nonzero vector $\mathbf{v}, \pm\frac{\mathbf{v}}{|\mathbf{v}|}$ are unit vectors parallel to $\mathbf{v}$. For a scalar $c > 0$, the vectors $\pm\frac{c\mathbf{v}}{|v|}$ are vectors of length c parallel to $\mathbf{v}$.

Properties of Vector Operations

Suppose $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

$$\mathbf{u}+a = \mathbf{v}+\mathbf{u} \qquad \text{Commutative property of addition} \tag{5}$$
$$(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w}) \qquad \text{Associative property of addition} \tag{6}$$
$$\mathbf{v}+\mathbf{0} = \mathbf{v} \qquad \text{Additive identity} \tag{7}$$
$$\mathbf{v}+(-\mathbf{v}) = 0 \qquad \text{Additive identity} \tag{8}$$
$$c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v} \qquad \text{Distributive property 1} \tag{9}$$
$$(a+c)\mathbf{v} = a\mathbf{v}+c\mathbf{v} \qquad \text{Distributive property 2} \tag{10}$$
$$0\mathbf{v} = \mathbf{0} \qquad \text{Multiplication by zero scalar} \tag{11}$$
$$c\mathbf{0} = \mathbf{0} \qquad \text{Multiplication by zero vector} \tag{12}$$
$$1\mathbf{v} = \mathbf{v} \qquad \text{Multiplicative identity} \tag{13}$$
$$a(c\mathbf{v}) = (ac)\mathbf{v} \qquad \text{Associative property of scalar multiplication} \tag{14}$$

12.2 Vectors in Three Dimensions

Distance Formula in xyz-Space

The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{15}$$

Spheres and Balls

A **sphere** centered at (a, b, c) with radius r is the set of points satisfying the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2 \tag{16}$$

A ball centered at (a, b, c) with radius r is the set of points satisfying the inequality

$$(x - a)^2 + (y - b)^2 + (z - c)^2 \leq r^2 \tag{17}$$

Vector Operations in $\mathbb{R}$

Let c be a scalar, $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$.

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle \qquad \text{Vector addition} \tag{18}$$

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3\rangle \qquad \text{Vector subtraction} \tag{19}$$

$$c\mathbf{u} = \langle cu_1, cu_2, c_3\rangle \tag{20}$$

Magnitude of a Vector

The **magnitude** (or **length**) of the vector $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$ is the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{21}$$

12.3 Dot Product

Dot Product

Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in two or three dimensions, their **dot product** is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta \tag{22}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ with $0 \leq \theta \leq \pi$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$, and θ is undefined.

Orthogonal Vectors

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. The zero vector is orthogonal to all vectors. In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.

Dot Product

Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{23}$$

Properties of the Dot Product

Suppose $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and let c be a scalar.

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \qquad \text{Commutative property} \tag{24}$$

$$c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) \qquad \text{Associative property} \tag{25}$$

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \tag{26}$$

(Orthogonal) Projection of u onto v

The **orthogonal projection of u onto v** , denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$, where $\mathbf{v} \neq \mathbf{0}$, is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right) \tag{27}$$

The orthogonal projection may also be computed with the formulas

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \text{scal}_{\mathbf{v}}\mathbf{u}\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} \tag{28}$$

where the **scalar component of u in the direction of v** is

$$\text{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} \tag{29}$$

Work

Let a constant force $\mathbf{F}$ be applied to an object, producing a displacement $\mathbf{d}$. If the angle between $\mathbf{F}$ and $\mathbf{d}$ is θ, then the **work** done by the force is

$$W = |\mathbf{F}||\mathbf{d}|\cos\theta = \mathbf{F}\cdot\mathbf{d} \tag{30}$$

12.4 Cross Product

Given two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector with magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta \tag{31}$$

where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$. The direction of $\mathbf{u} \times \mathbf{v}$ is given by the **right-hand rule**: When you put the vectors tail to tail and let the fingers of your right hand curl from $\mathbf{u}$ to $\mathbf{v}$ the direction of $\mathbf{u} \times \mathbf{v}$ is the direction of your thumb, orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. When $\mathbf{u} \times \mathbf{v} = 0$, the direction of $\mathbf{u} \times \mathbf{v}$ undefined.

Geometry of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbb{R}^3$.

1. The vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \times \mathbf{v} = 0$.

2. If $\mathbf{u}$ and $\mathbf{v}$ are two sides of a parallelogram, then the area of the parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta \tag{32}$$

Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^3$, and let a and b be scalars.

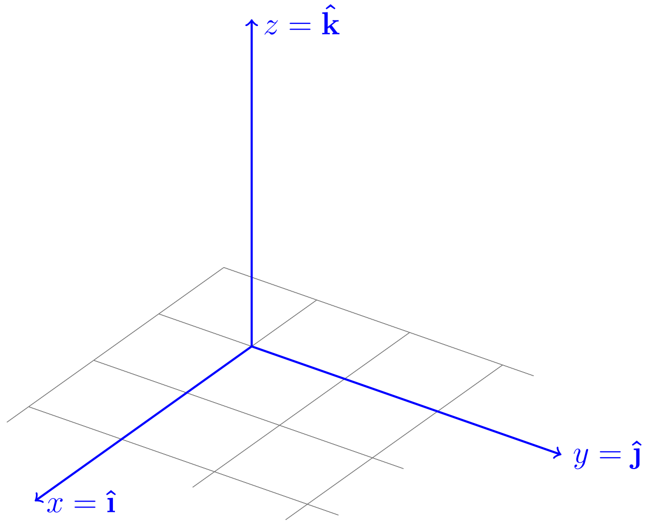
$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \quad \text{Anticommutative property} \tag{33}$$

$$(a\mathbf{u}) \times (b\mathbf{v}) = ab(\mathbf{u} \times \mathbf{v}) \quad \text{Associative property} \tag{34}$$

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) \quad \text{Distributive property} \tag{35}$$

$$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w}) \quad \text{Distributive property} \tag{36}$$

Cross Products of Coordinate Unit Vectors



$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = -(\hat{\mathbf{j}} \times \hat{\mathbf{i}}) = \hat{\mathbf{k}} \tag{37}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = -(\hat{\mathbf{k}} \times \hat{\mathbf{j}}) = \hat{\mathbf{i}} \tag{38}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = -(\hat{\mathbf{i}} \times \hat{\mathbf{k}}) = \hat{\mathbf{j}} \tag{39}$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0 \tag{40}$$

Evaluating the Cross Product

Let $\mathbf{u} = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}$ and $\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$. Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{\mathbf{k}} \tag{41}$$

12.5 Lines and Curves in Space

Equation of a Line

An equation of the line passing through the point $P_0(x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v} = \langle a, b, c\rangle$ is $\mathbf{r} = r_0 + t\mathbf{v}$, or

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle, \quad \text{for} \quad -\infty < t < \infty \tag{42}$$

Equivalently, the parametric equations of the line are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for} \quad -\infty < t < \infty \tag{43}$$

Limit of a Vector-Valued Function

A vector-valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written $\lim_{t\to a} \mathbf{r}(t) = \mathbf{L}$, provided $\lim_{t\to a} |\mathbf{r}(t) - \mathbf{L}| = 0$

12.6 Calculus of Vector-Valued Functions

Derivative and Tangent Vector

Let $\mathbf{r}(t) = f(t)\hat{\mathbf{\imath}} + g(t)\hat{\mathbf{\jmath}} + h(t)\hat{\mathbf{k}}$, where f, g, and h are differentiable functions on $(a\, b)$. Then $\mathbf{r}$ has a **derivative** (or is **differentiable**) on $(a\, b)$ and

$$\mathbf{r}'(t) = f'(i)\hat{\mathbf{\imath}} + g'(t)\hat{\mathbf{\jmath}} + h'(t)\hat{\mathbf{k}} \tag{44}$$

Provided $\mathbf{r}'(t) \neq \mathbf{0}, \mathbf{r}'(t)$ is a **tangent vector** (or velocity vector) at the point corresponding to $\mathbf{r}$.

Unit Tangent Vector

Let $\mathbf{r} = f(t)\hat{\mathbf{\imath}} + g(t)\hat{\mathbf{\jmath}} + h(t)\hat{\mathbf{k}}$ be a smooth parameterized curve, for $a \leq t \leq b$. The **unit tangent vector** for a particular value of t is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \tag{45}$$

Derivative Rules

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector-valued functions and let f be a differentiable scalar-valued function, all at a point t. Let $\mathbf{c}$ be a constant vector. The following rules apply.

$$\frac{d}{dt}(\mathbf{c}) = \mathbf{0} \quad \text{Constant Rule} \tag{46}$$

$$\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t) \quad \text{Sum Rule} \tag{47}$$

$$\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t) \quad \text{Product Rule} \tag{48}$$

$$\frac{d}{dt}(\mathbf{u}(f(t))) = \mathbf{u}'(f(t))f'(t) \quad \text{Chain Rule} \tag{49}$$

$$\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \quad \text{Dot Product Rule} \tag{50}$$

$$\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) \quad \text{Cross Product Rule} \tag{51}$$

Indefinite Integral of a Vector-Valued Function

Let $\mathbf{r} = f\hat{\mathbf{i}} + g\hat{\mathbf{j}} + h\hat{\mathbf{k}}$ be a vector function and let $\mathbf{R} = F\hat{\mathbf{i}} + G\hat{\mathbf{j}} + H\hat{\mathbf{k}}$, where F, G, and H are antiderivatives of f, g, and h, respectively. The **indefinite integral** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C} \tag{52}$$

where $\mathbf{C}$ is an arbitrary constant vector.

Definite Integral of a Vector-Valued Function

Let $\mathbf{r}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$, where f, g, and h are integrable on the interval $[a, b]$.

$$\int \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int_a^b g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int_a^b h(t)\,dt\right]\hat{\mathbf{k}} \tag{53}$$

12.7 Motion In Space

Position, Velocity, Speed, Acceleration

Let the position of an object moving in three-dimensional space be given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, for $t \geq 0$. The **velocity** of the object is

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle \tag{54}$$

The **speed** of the object is the scalar function

$$|\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \tag{55}$$

The **acceleration** of the object is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Motion with Constant $|\mathbf{r}|$

Let $\mathbf{r}$ describe a path on which $|\mathbf{r}|$ is constant (motion on a circle or sphere centered at the origin). Then, $\mathbf{r} \cdot \mathbf{v} = 0$, which means the position vector and the velocity vector are orthogonal at all times for which the functions are defined.

Two-Dimensional Motion in a Gravitational Field

Consider an object moving in a plane with horizontal x-axis and a vertical y-axis, subject only to the force of gravity. Given the initial velocity $\mathbf{v}(0) = \langle u_0,\, v_0\rangle$ and the initial position $\mathbf{r}(0) = \langle x_0,\, y_0\rangle$, the velocity of the object, for $t \geq 0$, is

$$\mathbf{v}(t) = \langle x'(t),\, y'(t)\rangle = \langle u_0,\, -gt + v_0\rangle \tag{56}$$

and the position is

$$\mathbf{r}(t) = \langle x(t),\, y(t)\rangle = \left\langle u_0\, t + x_0,\, -\frac{1}{2}gt^2 + v_0 t + y_0 \right\rangle \tag{57}$$

Two-Dimensional Motion

Assume an object traveling over horizontal ground, acted on only by the gravitational force, has an initial position $\langle x_0,\, y_0\rangle = \langle 0,\, 0\rangle$ and initial velocity

$\langle u_0, v_0\rangle = \langle |\mathbf{v}_0|\cos\alpha, |\mathbf{v}_0|\sin\alpha\rangle$. The trajectory, which is a segment or a parabola, has the following properties.

$$\text{time of flight} = T = \frac{2|\mathbf{v}_0|\sin\alpha}{g} \tag{58}$$

$$\text{range} = \frac{|\mathbf{v}_0|\sin 2\alpha}{g} \tag{59}$$

$$\text{maximum height} = y\left(\frac{T}{2}\right) = \frac{(|\mathbf{v}_0|\sin\alpha)^2}{2g} \tag{60}$$

12.8 Length of Curves

Consider the parameterized curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f', g', and h' are continuous, and the curve is traversed once for $a \leq t \leq b$. The **arc length** of the curve between $(f(a), g(a), h(a))$ and $(f(b), g(b), h(b))$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt = \int_a^b |\mathbf{r}'(\mathbf{t})|\, dt \tag{61}$$

Arc Length of a Polar Curve

Let f have a continuous derivative on the interval $[\alpha, \beta]$. The **arc length** of the polar curve $r = f(\theta)$ on $[\alpha,\ \beta]$ is

$$L = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta. \tag{62}$$

Arc Length as a Function of a Parameter

Let $\mathbf{r}$(t) describe a smooth curve, for $t \geq a$. The arc length is given by

$$s(t) = \int_a^t |\mathbf{v}(u)|\, du, \tag{63}$$

where $|\mathbf{v}| = |\mathbf{r}'|$. Equivalently, $\frac{ds}{dt} = |\mathbf{v}t| > 0$. If $|\mathbf{v}(t)| = 1$, for all $t \geq a$, then the parameter t corresponds to arc length.

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12.9 Curvature and Normal Vectors

Curvature

Let $\mathbf{r}$ describe a smooth parameterized curve. If s denotes arc length and $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$ is the unit tangent vector, the **curvature** is $\kappa(s) = |\frac{d\mathbf{T}}{ds}|$

Curvature Formula

Let $\mathbf{r}(t)$ describes a smooth parameterized curve, where t is any parameter. If $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{T}$ is the unit tangent vector, then the curvature is

$$\kappa(t) = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \left|\frac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right| \tag{65}$$

Alternative Curvature Formula

Let $\mathbf{r}$ be the position of an object moving on a smooth curve. The **curvature** at a point on the curve is

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}, \tag{66}$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity and $\mathbf{a} = \mathbf{v}'$ is the acceleration.

Principal Unit Normal Vector

Let $\mathbf{r}$ describe a smooth parameterized curve. The **principal unit normal vector** at a point P on the curve at which $\kappa \neq 0$ is

$$\mathbf{N}(s) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} \tag{67}$$

In practice, we use the equivalent formula

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \tag{68}$$

evaluated at the value of t corresponding to P.

Properties of the Principal Unit Normal Vector

Let $\mathbf{r}$ describe a smooth parameterized curve with unit tangent vector $\mathbf{T}$ and principal unit normal vector $\mathbf{N}$.

1. $\mathbf{T}$ and $\mathbf{N}$ are orthogonal at all points of the curve; that is, $\mathbf{T}(t)\cdot\mathbf{N}(t) = 0$, at all points where $\mathbf{N}$ is defined.

2. The principal unit normal vector points to the inside of the curve—in the direction that the curve is turning.

Tangential and Normal Components of the Acceleration

The acceleration vector of an object moving in space along a smooth curve has the following representation in terms of its **tangential component** a_T (in the direction of $\mathbf{T}$) and its normal component a_N (in the direction of $\mathbf{N}$):

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}, \tag{69}$$

where $a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|}$ and $a_T = \frac{d^2s}{dt^2}$.

Unit Binormal Vector and Torsion

Let C be a smooth parameterized curve with unit tangent and principal unit normal vectors $\mathbf{T}$ and $\mathbf{N}$, respectively. Then, at each point of the curve at which the curvature is nonzero, the **unit binormal vector** is

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} \tag{70}$$

and the **torsion** is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} \tag{71}$$

Formulas for Curves in Space

1. Position function: $\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$

2. Velocity: $\mathbf{v} = \mathbf{r}'$

3. Acceleration: $\mathbf{a} = \mathbf{v}'$

4. Unit tangent vector: $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

5. Principal unit normal vector: $\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$ (provided $d\mathbf{T}/dt \neq \mathbf{0}$)

6. Curvature: $\kappa = \frac{d\mathbf{T}}{ds} = \frac{1}{|\mathbf{v}|}|\frac{d\mathbf{T}}{dt}| = |\frac{\mathbf{T}'(t)}{\mathbf{r}'(t)}| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$

7. Components of acceleration: $\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}$, where $a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|}$ and $a_T = \frac{d^2s}{dt^2} = \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}|}$

8. Unit binormal vector: $\mathbf{B} = \mathbf{B} \times \mathbf{N} = \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|}$

9. Torsion $\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \frac{(\mathbf{v}\times\mathbf{a})\cdot\mathbf{a}'}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{(\mathbf{r}'\times\mathbf{r}'')^2}$

13 Functions of Several Variables

13.1 Planes and Surfaces

Plane in $\mathbb{R}^3$

Given a fixed point P_0 and a nonzero **normal vector n**, the set of points P in $\mathbb{R}^3$ for which $\vec{P_0P}$ is orthogonal to **n** is called a **plane**.

General Equation of a Plane in $\mathbb{R}^3$

The plane passing through the point $P_0(x_0, y_0, z_0)$ with a nonzero normal vector $\mathbf{n} = \langle a, b, c\rangle$ is described by the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d \tag{72}$$

where $d = ax_0 + by_0 + cz_0$.

Parallel and Orthogonal Planes

Two distinct planes are **parallel** if their respective normal vectors are parallel (that is, the normal vectors are scalar multiples of each other). Two planes are **orthogonal** if their respective normal vectors are orthogonal (that is, the dot product of the normal vectors is zero).

Cylinder

Given a curve C in a plane P and a line ℓ not in P, a **cylinder** is the surface consisting of all lines parallel to ℓ that pass through C.

Trace

A **trace** of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes. The trace in the coordinate planes are called **xy-trace**, the **xz-trace**, and the **yz-trace**.

Quadratic Surfaces

Name	Standard Equation	Features
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	All traces are ellipses.
Elliptic paraboloid	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	Traces with $z = z_0 > 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are parabolas.
Hyperboloid of one sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Traces with $z = z_0$ are ellipses for all z_0. Traces with $x = x_0$ or $y = y_0$ are hyperbolas.
Hyperboloid of two sheets	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Traces with $z = z_0$ with $\|z_0\| > \|c\|$ are ellipses. Traces with $x = x_0$ and $y = y_0$ are hyperbolas.
Elliptic cone	$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$	Traces with $z = z_0 \neq 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are hyperbolas or intersecting lines.
Hyperbolic paraboloid	$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$	Traces with $z = z_0 \neq 0$ are hyperbolas. Traces with $x = x_0$ or $y = y_0$ are parabolas.

13.2 Graphs and Level Curves

Function, Domain, and Range with Two Independent Variables

A **function** $z = f(x, y)$ assigns to each point (x, y) in a set D in $\mathbb{R}^2$ a unique real number z in a subset of $\mathbb{R}$. The set D is the **domain** of f. The **range** of f is the set of real numbers z that are assumed as the points (x, y) vary over the domain.

Function, Domain, and Range with n Independent Variables

The **function** $y = f(x_1, x_2, \ldots, x_n)$ assigns a unique real number y to each point $(x_1, x_2, \ldots, x_n)$ in a set D in $\mathbb{R}^n$. The set D is the **domain** of f. The **range** is the set of real numbers y that are assumed as the points $(x_1, x_2, \ldots, x_n)$ vary over the domain.

13.3 Limits and Continuity

Limit of a Function of Two Variables

The function f has the **limit** L as $P(x, y)$ approaches $P_0(a, b)$ written

$$\lim (x, y) \to (a, b) f(x, y) = \lim P \to P_0 f(x, y) = L \tag{73}$$

if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \tag{74}$$

whenever (x, y) is in the domain of f and

$$0 < |PP_0| = \sqrt{(x-a)^2 + (y-b)^2} < \delta \tag{75}$$

Limits of Constant and Linear Functions

Let a, b, and c be real numbers.

1. Constant functions $f(x, y) = c : \lim_{(x, y)\to(a, b)} c = c$
2. Linear function $f(x, y) = x : \lim_{(x, y)\to(a, b)} x = a$
3. Linear function $f(x, y) = y : \lim_{(x, y)\to(a, b)} y = b$

Limit Laws and Functions of Two Variables

Let L and M be real numbers and suppose that $\lim_{(x, y)\to(a, b)} f(x, y) = L$ and $\lim_{(x, y)\to(a, b)} g(x, y) = M$. Assume c is a constant, and $\forall m, n \in \mathbb{Z}$.

1 Sum $\lim_{(x, y)\to(a, b)} (f(x, y) + g(x, y)) = L + M$

2 Difference $\lim_{(x, y)\to(a, b)} (f(x, y) - g(x, y)) = L - M$

3 Constant multiple $\lim_{(x, y)\to(a, b)} cf(x, y) = cL$

4 Product $\lim_{(x, y)\to(a, b)} f(x, y) \cdot g(x, y) = L \cdot M$

5 Quotient $\lim_{(x, y)\to(a, b)} \left[\frac{f(x, y)}{g(x, y)}\right] = \frac{L}{M}$

6 Power $\lim_{(x,\,y)\to(a,\,b)} (f(x,\,y))^n = L^n$

7 $^m/_n$ Power If m and n have no common factors and $n \neq 0$, then $\lim_{(x,\,y)\to(a,\,b)} [f(x,\,y)]^{m/n} = L^{m/n}$, where we assume $L > 0$ if n is even.

Interior and Boundary Points

Let R be a region in $\mathbb{R}^2$. An **interior point** P of R lies entirely within R, which means it is possible to find a disk centered at P that contains only points of R.
A **boundary point** Q of R lies on the edge of R in the sense that *every* disk centered at Q contains at least one point in R and at least one point not in R.

Open and Closed Sets

A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points.

Two-Path Test for Nonexistence of Limits

If $f(x,\,y)$ approaches two different values as $(x,\,y)$ approaches $(a,\,b)$ along two different paths in the domain of f, then $\lim_{(x,\,y)\to(a,\,b)} f(x,\,y)$ does not exist.

Continuity

A function f is continuous at the point $(a,\,b)$ provided

1. f is defined at $(a,\,b)$.
2. $\lim_{(x,\,y)\to(a,\,b)} f(x,\,y)$ exists.
3. $\lim_{(x,\,y)\to(a,\,b)} f(x,\,y) = f(a,\,b)$.

Continuity of Composite Functions

If $u = g(x,\,y)$ is continuous at $(a,\,b)$ and $z = f(u)$ is continuous at $g(a,\,b)$, then the composite function $z = f(g(x,\,y))$ is continuous at $(a,\,b)$.

13.4 Partial Derivatives

The **partial derivative of f with respect to x at the point (a, b)** is

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}. \tag{76}$$

The **partial derivative of f with respect to y at the point (a, b)** is

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h}. \tag{77}$$

provided these limits exists.

Equality of Mixed Partial Derivatives

Assume that f is defined on an open set D of $\mathbb{R}^2$, and f_{xy} and f_{yx} are continuous throughout D. Then $f_{xy} = f_{yx}$ at all points of D.

Differentiability

The function $z = f(x,\, y)$ is **differentiable at** $(a,\, b)$ provided $f_x(a,\, b)$ and $f_y(a,\, b)$ exist and the change $\Delta z = f(a + \Delta x,\, b + \Delta y) - f(a,\, b)$ equals

$$\Delta z = f_x(a,\, b)\Delta x + f_y(a,\, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y, \tag{78}$$

where for fixed a and b, ε_1 and ε_2 are functions that depend only on Δx and Δy, with $(\varepsilon_1, \varepsilon_2) \to (0,\, 0)$ as $(\Delta x,\, \Delta y) \to (0,\, 0)$. A function is **differentiable** on an open set R if it is differentiable at every point on R.

Conditions for Differentiability

Suppose the function f has partial derivatives f_x and f_y defined on an open set containing $(a,\, b)$, with f_x and f_y continuous $(a,\, b)$. Then f is differentiable at $(a,\, b)$.

Differentiability Implies Continuity

If a function f is differentiable at $(a,\, b)$, then it is continuous at $(a,\, b)$

13.5 The Chain Rule

Chain Rule (One Independent Variable)

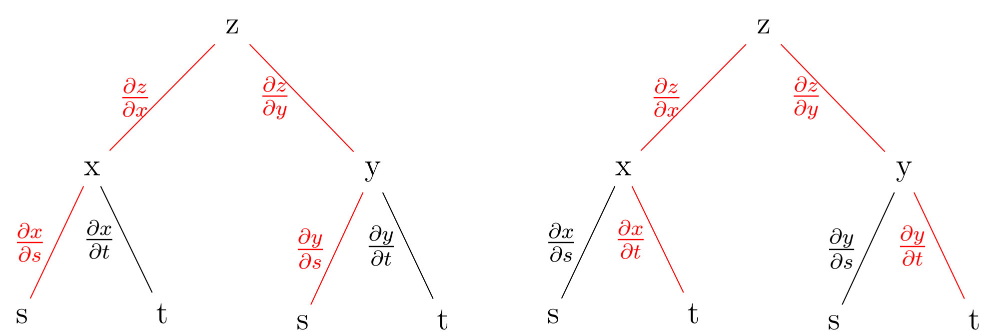
Let z be a differentiable function of x and y on its domain, where x and y are differentiable functions of t on an interval I. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{79}$$

Chain Rule (Two Independent Variables)

Let z be a differentiable function of x and y, where x and y are differentiable functions of s and t. Then

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \tag{80}$$



Implicit Differentiation

Let F be differentiable on its domain and suppose that $F(x, y) = 0$ defines y as a differentiable function of x. Provided $F_y \neq 0$.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \tag{81}$$

13.6 Directional Derivatives and the Gradient

Directional Derivative

Let f be a differentiable at $(a,\, b)$ and let $\mathbf{u} = \langle \cos\theta,\, \sin\theta\rangle$ be a unit vector in the xy-plane. The **directional derivatives of f at** $(a,\, b)$ in the direction of $\mathbf{u}$ is

$$D_u f(a,\, b) = \lim_{h\to 0} \frac{f(a + h\cos\theta,\, b + h\sin\theta) - f(a,\, b)}{h} \tag{82}$$

provided the limit exits.

Directional Derivative

Let f be differentiable on $(a,\, b)$ and let $\mathbf{u} = \langle u_1,\, u_2\rangle$ be a unit vecor in the $x,\, y$-plane. The **directional derivative of f at a** $(a,\, b)$ **in the direction of** u is

$$D_u f(a,\, b) = \langle f_x(a,\, b),\, f_y(a,\, b)\rangle \cdot \langle u_1,\, u_2\rangle \tag{83}$$

Gradient (Two Dimensions)

Let f be differentiable at the point $(x,\, y)$. The **gradient** of f at $(x,\, y)$ is the vector-valued function

$$\nabla f(x,\, y) = \langle f_x(x,\, y),\, f_y(x,\, y)\rangle = f_x(x,\, y)\hat{\mathbf{\imath}} + f_y(x,\, y)\hat{\mathbf{\jmath}} \tag{84}$$

Directions of Change

Let f be differentiable at $(a,\, b)$ with $\nabla f(a,\, b) \neq \mathbf{0}$

1. f has its maximum rate of increase at $(a,\, b)$ in the direction of the gradient $\nabla f(a,\, b)$. The rate of increase in this direction is $|\nabla f(a,\, b)|$
2. f has its maximum rate of decrease at $(a,\, b)$ in the direction of $-\nabla f(a,\, b)$. The rate of decrease in this direction is $-|\nabla f(a,\, b)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a,\, b)$.

The Gradient and Level Curves

Given a function f differentiable at $(a,\, b)$, the line tangent to the level curve of f at $(a,\, b)$ is orthogonal to the gradient $\nabla f(a,\, b)$, provided $\nabla f(a,\, b) \neq \mathbf{0}$.

Gradient and Directional Derivative in Three Dimensions

Let f be differentiable at the point $(x,\, y,\, z)$. The **gradient** of f at $(x,\, y,\, z)$ is the vector-valued function

$$\begin{aligned}\nabla f(x,\, y,\, z) &= \langle f_x(x,\, y,\, z),\, f_y(x,\, y,\, z),\, f_z(x,\, y,\, z)\rangle && (85)\\ &= f_x(x,\, y,\, z)\hat{\mathbf{i}} + f_y(x,\, y,\, z)\hat{\mathbf{j}} + f_z(x,\, y,\, z)\hat{\mathbf{k}} && (86)\end{aligned}$$

The **directional derivative** of f in the direction of the unit vector $\mathbf{u} = \langle u_1,\, u_2,\, u_3\rangle$ at the point $(a,\, b,\, c)$ is $D_u f(a,\, b,\, c) = \nabla f(a,\, b,\, c) \cdot \mathbf{u}$

13.7 Tangent Planes and Linear Approximation

Let F be differentiable at the point $P_0(a, b, c)$ with $\nabla F(a, b, c) \neq \mathbf{0}$. The plane tangent to the surface $F(x, y, z) = 0$ at P_0, called the **tangent plane**, is the plane passing through P_0 orthogonal $\nabla F(a, b, c)$. An equation of the tangent plane is

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0. \tag{87}$$

Tangent Plane for $z = f(x, y)$

Let f be differentiable at the point (a, b). An equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b) \tag{88}$$

Linear Approximation

Let f be differentiable at (a, b). The linear approximation to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is the tangent plane at that point, given by the equation

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b) \tag{89}$$

The Differential dz

Let f be differentiable at the point (a, b). The change in $z = f(x, y)$ as the independent variables change from (a, b) to $(a + dx, b + dy)$ is denoted by the differential dz:

$$\Delta z \approx dz = f_x(a, b)dx + f_y(a, b)dy \tag{90}$$

13.8 Maximum/Minimum Problems

Local Maximum/Minimum Values

A function f has a **local maximum value** at (a, b) if $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f in some open disk centered at (a, b). A function f has a **local minimum value** at (a, b) if $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f in some open disk centered at (a, b). Local maximum and local minimum values are also called **local extreme values** or **local extrema**.

Derivatives and Local Maximum/Minimum Values

If f has a local maximum or minimum value at (a, b) and the partial derivatives f_x and f_y exist at (a, b) then $f_x(a, b) = f_y(a, b) = 0$.

Critical Point

An interior point $(a,\, b)$ in the domain of f is a **critical point** of f is either

1. $f_x(a,\, b) = f_y(a,\, b) = 0$, or
2. one (or both) of f_x or f_y does not exist at $(a,\, b)$

Saddle Point

A function f has a **saddle point** at a critical point $(a,\, b)$ if, in every open disk centered at $(a,\, b)$, there are points $(x,\, y)$ for which $f(x,\, y) > f(a,\, b)$ and points for which $f(x,\, y) < f(a,\, b)$

Second Derivative Test

Suppose that the second partial derivative of f are continuous throughout an open disk centered at the point $(a,\, b)$, where $f_x(a,\, b) = f_y(a,\, b) = 0$. Let $D(x,\, y) = f_{xx}(x,\, y) f_{yy}(x,\, y) - (f_{xy}(x,\, y))^2$.

1. If $D(a,\, b) > 0$ and $f_{xx}(a,\, b) < 0$, then f has a local maximum value at $(a,\, b)$.
2. If $D(a,\, b) > 0$ and $f_{xx}(a,\, b) > 0$, then f has a local minimum value at $(a,\, b)$.

3. If $D(a, b) < 0$, then f has a saddle point at (a, b).
4. If $D(a, b) = 0$, then the test is inconclusive.

Absolute Maximum/Minimum Values

If $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f, then f has an **absolute maximum value** at (a, b). If $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f, then f has an **absolute minimum value** at (a, b).

Finding Absolute Maximum/Minimum Values on Closed, Bounded Sets

Let f be continuous on a closed bounded set R in $\mathbb{R}^2$. To find the absolute maximum and minimum values of f on R:

1. Determine the values of f at all critical points in R.
2. Find the maximum and minimum values of f on the boundary of R.
3. The greatest function values found in Step 1 and 2 is the absolute maximum value of f on R, and the least function value found in Steps 1 and 2 is the absolute minimum value of f on R.

13.9 Lagrange Multipliers

Parallel Gradients (Ball Park Theorem)

Let f be a differentiable function in a region of $\mathbb{R}^2$ that contains the smooth curve C given by $g(x, y) = 0$. Assume that f has a local extreme value (relative to values of f on C) at a point $P(a, b)$ on C. Then $\nabla f(a, b)$ is orthogonal to the line tangent to C at P. Assuming $\nabla g(a, b) \neq 0$, it follows that there is a real number λ (called a **Lagrange multiplier**) such that $\nabla f(a, b) = \lambda \nabla g(a, b)$.

Method of Lagrange Multipliers in Two Variables

Let the objective function f and the constraint function g be differentiable on a region of $\mathbb{R}^2$ with $\lambda g(x, y) \neq 0$ on the curve $g(x, y) = 0$. To locate the maximum and minimum values of f subject to the constraint $g(x, y) = 0$, carry out the following steps.

1. Find the values of x, y and λ (if they exist) that satisfy the equations

$$\nabla f(x,\, y) = \lambda \nabla g(x,\, y) \quad \text{and} \quad g(x,\, y) = 0 \tag{91}$$

2. Among the values $(x,\, y)$ found in Step 1, select the largest and smallest corresponding function values, which are the maximum and minimum values of f subject to the constraint.

Method of Lagrange Multipliers in Three Variables

Let f and g be differentiable on a region of $\mathbb{R}^3$ with $\nabla g(x,\, y,\, z) neq 0$ on the surface $g(x,\, y,\, z) = 0$. To locate the maximum and minimum values of f subject to the constraint $g(x,\, y,\, z) = 0$, carry out the following steps.

1. Find the values of $x,\, y,\, z$ and λ that satisfy the equations

$$\nabla f(x,\, y,\, z) = \lambda \nabla g(x,\, y,\, z) \quad \text{and} \quad g(x,\, y,\, z) = 0 \tag{92}$$

2. Among the points $(x,\, y,\, z)$ found in Step 1, select the largest and smallest corresponding values of the objective function. These values are the maximum and minimum values of f subject to the constraint.

14 Multiple Integration

14.1 Double Integrals over Rectangular Regions

Volumes and Double Integrals

A function f defined on a rectangular region R in the xy-plane is **integratabtle** on R if $\lim\limits_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$ exists fora ll partitions of R and for all choices of (x_k^*, y_k^*) within those partitions. The limit is the **double integral of** f **over** R, which we write

$$\iint\limits_R f(x, y)\, dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \tag{93}$$

If f is nonnegative on R, then the double integral equals the volume of the solid bounded by $z = f(x, y)$ and the xy-plane over R.

Double Integrals on Rectangular Regions

Let f be continuous on the rectangular region $R = \{(x, y) : a \leq x \leq b,\, c \leq y \leq d\}$. The double integral of f over R may be evaluated by either of two iterated integrals:

$$\iint\limits_R f(x, y)\, dA = \int_c^d \int_a^b dx\, dy = \int_a^b \int_c^d dy\, dx \tag{94}$$

Average Value of a Function over a Plane Region

The **average value** of an integrable function f over a region R is

$$\bar{f} = \frac{1}{\text{area of } R} \iint\limits_R f(x, y)\, dA \tag{95}$$

14.2 Double Integrals over General Regions

Let R be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines $x = a$ and $x = b$. If f is continuous on R, then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx \tag{96}$$

Let R be a region bounded on the left and right by the graphs of the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and the lines $y = c$ and $y = d$. If f is continuous on R, then

$$\iint\limits_R f(x,\, y)\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x,\, y)\, dx\, dy \tag{97}$$

Areas of Regions by Double Integrals

Let R be a region in the xy-plane. Then

$$\text{area of } R = \iint\limits_R \tag{98}$$

14.3 Double Integrals in Polar Coordinates

Double Integrals over Polar Rectangular Region

Let f be continuous on the region in the xy-plane $R = \{(r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$, where $\beta - \alpha \leq 2\pi$. Then

$$\iint\limits_R f(r, \theta)\, dA = \int_\alpha^\beta \int_a^b f(r, \theta)\ r\, dr\, d\theta \tag{99}$$

Double Integrals over More General Polar Regions

Let f be continuous on the region in the xy-plane

$$R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\} \tag{100}$$

where $0 < \beta - \alpha \leq 2\pi$. Then

$$\iint\limits_R f(r, \theta)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r, \theta) r\, dr\, d\theta. \tag{101}$$

Area of Polar Regions

The area of the region $R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\}$, where $0 < \beta - \alpha \leq 2\pi$, is

$$A = \iint\limits_R dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r\, dr\, d\theta \tag{102}$$

14.4 Triple Integrals

Let f be continuous over the region

$$D = \{(x,\, y,\, z) : a \leq x \leq b, g(x) \leq y \leq h(x),\, G(x,\, y) \leq z \leq H(x,\, y)\} \tag{103}$$

where g, h, G, and H are continuous functions. Then f is integrable over D and the triple integral is evaluated as the iterated integral

$$\iiint\limits_D f(x,\, y,\, z)\, dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x,\, y,\, z)\, dz\, dy\, dx. \tag{104}$$

Average Value of a Function of Three Variables

If f is continuous on a region D of $\mathbb{R}^3$, then the average value of over D is

$$\bar{f} = \frac{1}{\text{volume } (D)} \iiint\limits_D f(x,\, y,\, z)\, dV \tag{105}$$

14.5 Triple Integrals in Cylindrical and Spherical Coordinates

Transformations Between Cylindrical and Rectangular Coordinates

Rectangular → Cylindrical	**Cylindrical → Rectangular**
$r^2 = x^2 + y^2$	$x = r\cos\theta$
$\tan\theta = {}^{y}\!/_{x}$	$y = r\sin\theta$
$z = z$	$z = z$

Triple Integrals in Cylindrical Coordinates

Let f be continuous over the region

$$D = \{(r,\, \theta,\, z) : g(\theta) \le r \le h(\theta),\, \alpha \le \theta \le \beta,\, G(x,\, y) \le z \le H(x,\, y).\} \tag{106}$$

Then f is integrable over D and the triple integral of f over D in cylindrical coordinates is

$$\iiint\limits_D f(r,\, \theta,\, z)\, dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta,\, r\sin\theta)}^{H(r\cos\theta,\, r\sin\theta)} f(r,\, \theta,\, z)\, dz\; r\, dr\, d\theta \tag{107}$$

Tranformations Between Spherical and Rectangular Coordinates

Rectangular → Spherical	**Spherical → Rectangular**
$\rho^2 = x^2 + y^2 + z^2$	$x = \rho\sin\varphi\cos\theta$
Use trigonometry to find φ and θ	$y = \rho\sin\varphi\sin\theta$
	$z = \rho\cos\varphi$

Triple Integrals in Spherical Coordinates

Let f be continuous over the region

$$D = \{(\rho,\, \varphi,\, \theta) : g(\varphi,\, \theta) \le \rho \le h(\varphi,\, \theta),\, a \le \varphi b,\, \alpha \le \theta \le \beta\} \tag{108}$$

Then f is integrable over D, and the triple integral of f over D in spherical coordinates is

$$\iiint\limits_D f(\rho,\, \varphi,\, \theta) dV = \int_\alpha^\beta \int_a^b \int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho,\, \varphi,\, \theta)\rho^2 \sin\varphi \; d\rho\, d\varphi\, d\theta \tag{109}$$

14.6 Integrals for Mass Calculations

Center of Mass in One Dimension

Let ρ be an integrable density function on the interval $[a, b]$ (which represents a thin rod or wire). The **center of mass** is location at the point $\bar{x} = \frac{M}{m}$, where the **total moment** M and mass m are

$$M = \int_a^b x\rho(x)\,dx \quad \text{and} \quad m = \int_a^b \rho(x)\,dx \tag{110}$$

Center of Mass in Two Dimensions

Let ρ be integrable density function defined over a closed bounded region R in $\mathbb{R}^2$. The coordinates of the center of mass of the object represented by R are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint\limits_R x\rho(x,\, y)\,dA \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m} = \iint\limits_R y\rho(x,\, y)\,dA \tag{111}$$

Center of Mass in Three Dimensions

Let ρ be integrable density function on a closed bounded region D in $\mathbb{R}^3$. The coordinates of the center of mass of the region are

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m}\iiint\limits_R x\rho(x,\, y,\, z)\,dV \tag{112}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1}{m}\iiint\limits_R y\rho(x,\, y,\, z)\,dV \tag{113}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\iiint\limits_R z\rho(x,\, y,\, z)\,dV \tag{114}$$

where $m = \iiint_D \rho(x,\, y,\, z)\,dV$ is the mass, and M_{yz}, M_{xz} and M_{xy} are the moments with respect to the coordinate planes.

15 Vector Calculus

15.1 Vector Fields

Vector Fields in Two Dimensions

Let f and g be defined on a region R of $\mathbb{R}^2$. A **vector field** in $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point in R a vector $\langle f(x,y),\, g(x,y)\rangle$. The vector field is written as

$$\mathbf{F}(x,y) = \langle f(x,y),\, g(x,y)\rangle \quad \text{or} \quad \mathbf{F}(x,y) = f(x,y)\hat{\mathbf{\imath}} + g(x,y)\hat{\mathbf{\jmath}} \tag{115}$$

A vector field $\mathbf{F} = \langle f, g\rangle$ is continuous or differentiable on a region R of $\mathbb{R}^2$ if f and g are continuous or differentiable on R, respectively.

Radial Vector Fields in $\mathbb{R}^2$

Let $\mathbf{r} = \langle x,\, y\rangle$. A vector field of the form $\mathbf{F} = f(x,\, y)\mathbf{r}$, where f is a scalar-valued function, is a **radial vector field**. Of specific interest are the radial vector field

$$\mathbf{F}(x,\, y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x,\, y\rangle}{|\mathbf{r}|^p} \tag{116}$$

where p is a real number. At every point (except the origin), the vectors of this field are directed outward from the origin with the magnitude of $|\mathbf{F}| = \frac{1}{|\mathbf{r}|^{p-1}}$.

Vector Fields in Three Dimensions

Let $f,\ g,$ and h be defined on a region D of $\mathbb{R}^3$. A **vector field** in $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point in D a vector $\langle f(x,\, y,\, z),\, g(x,\, y,\, z),\, h(x,\, y,\, z)\rangle$. The vector field is written as

$$\mathbf{F}(x,\, y,\, z) = \langle f(x,\, y,\, z),\, g(x,\, y,\, z),\, h(x,\, y,\, z)\rangle \quad \text{or} \tag{117}$$

$$\mathbf{F}(x,\, y,\, z) = f(x,\, y,\, z)\hat{\mathbf{\imath}} + g(x,\, y,\, z)\hat{\mathbf{\jmath}} + h(x,\, y,\, z)\hat{\mathbf{k}} \tag{118}$$

A vector field $\mathbf{F} = \langle f,\, g,\, h\rangle$ is continuous or differentiable on a region D of $\mathbb{R}^3$ if $f,\ g,\ h$ are continuous or differentiable on R, respectively. Of particular importance are the **radial vector fields**

$$\mathbf{F}(x,\, y,\, z) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x,\, y,\, z\rangle}{|\mathbf{r}|^p} \tag{119}$$

where p is a real number.

Gradient Fields and Potential Functions

Let $z = \varphi(x,\, y)$ and $w = \varphi(x,\, y,\, z)$ be differentiable functions on regions of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. The vector field $\mathbf{F} = \nabla\varphi$ is **gradient field**, and the function φ is a **potential function** for $\mathbf{F}$.

15.2 Line Integrals

Scalar Line Integral in the Plane, Arc Length Parameter

Suppose the scalar-valued function f is defined on the smooth curve C : $\mathbf{r}(s) = \langle x(s),\, y(s)\rangle$, parameterized by the arc length s. The **line integral of** f **over** C is

$$\int\limits_C f(x(s),\, y(s))\, ds = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x(s_k^*),\, y(s_k^*))\Delta s_k, \tag{120}$$

provided this limit exists over all partitions of C. When the limit exists, f is said to be **integrable** on C.

Evaluating Scalar Line Integrals in $\mathbb{R}^2$

Let f be continuous on a region containing a smooth curve C : $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \leq t \leq b$. Then

$$\int\limits_C f\, ds = \int_a^b f(x(t),\, y(t))|\mathbf{r}'(t)|\, dt \tag{121}$$

$$= \int_a^b f(x(t),\, y(t))\sqrt{x'(t)^2 + y'(t)^2}\, dt \tag{122}$$

Evaluating the Line Integral $\int\limits_C f\, ds$

1. Find a parametric description of C in the form $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \leq t \leq b$
2. Computer $|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$
3. Make substitutions for x and y in the integrand and evaluate an ordinary integral

$$\int\limits_C f\, ds = \int_a^b f(x(t),\, y(t))|\mathbf{r}'(t)|\, dt \tag{123}$$

Evaluating Scalar Line Integrals in $\mathbb{R}^3$

Let f be continuous on a region containing a smooth curve $C : \mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$, for $a \le t \le b$. Then

$$\int f\,ds = \int_a^b f(x(t),\, y(t),\, z(t))|\mathbf{r}'(t)|\,dt \tag{124}$$

$$= \int_a^b f(x(t),\, y(t),\, z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt \tag{125}$$

Line Integral of a Vector Field

Let $\mathbf{F}$ be a vector field that is continuous on a region containing a smooth oriented curve C parameterized by arc length. Let $\mathbf{T}$ be the unit tangent vector at each point of C consistent with the orientation. The line integral of $\mathbf{F}$ over C is $\int_C \mathbf{F} \cdot \mathbf{T}\,ds$.

Different Forms of Line Integrals of Vector Fields

The line integral $\int_C \mathbf{F} \cdot \mathbf{T}\,ds$ may be expressed in the following forms, where $\mathbf{F} = \langle f,\, g,\, h\rangle$, for $a \le t \le b$:

$$\int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\,dt = \int_a^b (f\,x'(t),\, g\,y'(t),\, h\,z'(t))\,dt \tag{126}$$

$$= \int_C f\,dx + g\,dy + h\,dz \tag{127}$$

$$= \int_C \mathbf{F} \cdot d\mathbf{r} \tag{128}$$

For line integrals in the plane, we let $\mathbf{F} = \langle f,\, g\rangle$ and assume C is parameterized in the form $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$. Then

$$\int_C \mathbf{F} \cdot \mathbf{T}\,ds = \int_a^b (f\,x'(t) + g\,y'(t))\,dt = \int_C f\,dx + g\,dy = \int_C \mathbf{F} \cdot d\mathbf{r} \tag{129}$$

Work Done in a Force Field

Let $\mathbf{F}$ be a continuous force field in a region D of $\mathbb{R}^3$ and let $C : \mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$, for $a \le t \le b$, be a smooth curve in D with a unit tangent vector $\mathbf{T}$ consistent with the orientation. The work done in moving an object C in the positive direction is

$$W = \int\limits_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(\mathbf{t})\, dt \tag{130}$$

Circulation

Let $\mathbf{F}$ be a continuous vector field on a region D of $\mathbb{R}^3$ and let C be a closed smooth oriented curve in D. The **circulation** of $\mathbf{F}$ on C is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{T}$ is the unit vector tangent to C consistent with the orientation.

Flux

Let $F = \langle f, g\rangle$ be continuous vector field on a region R of $\mathbb{R}^2$. Let $C : \mathbf{r}(t) = \langle x(t),\, y(t)\rangle$, for $a \le t \le b$, be a smooth oriented curve in R that does not intersect itself. The **flux** of the vector field across C is

$$\int\limits_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b (f\, y'(t) - g\, x'(t))\, dt, \tag{131}$$

where $n = \mathbf{T} \times \hat{\mathbf{k}}$ is the unit normal vector and $\mathbf{T}$ is the unit tangent vector consistent with the orientation. If C is a closed curve with counterclockwise orientation, $\mathbf{n}$ is the outward normal vector and the flux integral gives the **outward flux** across C.

15.3 Conservative Vector Fields

Simple and Closed Curves

Suppose a curve C (in $\mathbb{R}^2$ and $\mathbb{R}^3$) is described parametrically by $\mathbf{r}(t)$, where $a \le t \le b$. Then C is a **simple curve** if $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ for all t_1 and t_2, with $a < t_1 < t_2 < b$; that is, C never intersects itself between its endpoints. The curve C is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$; that is, the initial and terminal points of C are the same.

Connected and Simply Connected Regions

An open region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is **connected** if it possible to connect any two points of R by a continuous curve lying in R. An open region R is **simply connected** if every closed simple curve in R can be deformed and contracted to a point in R.

Conservative Vector Field

A vector field F is said to be **conservative** on a region (in $\mathbb{R}^2$ or $\mathbb{R}^3$) if there exists a scalar function φ such that $\mathbf{F} = \nabla\varphi$ on that region.

Test for Conservative Vector Fields

Let $\mathbf{F} = \langle f,\, g,\, h\rangle$ be a vector field defined on a connected and simply connected region D of $\mathbb{R}^3$, where f, g, and h have continuous first partial derivatives on D. Then $\mathbf{F}$ is a conservative vector field on D (there is a potential function φ such that $\mathbf{F} = \nabla\varphi$) if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \text{and} \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y} \tag{132}$$

For vector fields in $\mathbb{R}^2$, we have the single condition $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

Finding Potential Functions in $\mathbb{R}^3$

Suppose $\mathbf{F} = \langle f,\, g,\, h\rangle$ is a conservative vector field. To find φ such that $\mathbf{F} = \nabla\varphi$, take the following steps:

1. Integral $\varphi_x = f$ with respect to x to obtain φ, which includes an arbitrary function $c(y,\, z)$.

2. Compute φ_y and equate it to g to obtain an expression for $c_y(y,\, z)$.
3. Integrate $c_y(y,\, z)$ with respect to y to obtain $c(y,\, z)$, including an arbitrary function $d(z)$.
4. Compute φ_z and equate it to h to get $d(z)$.

Beginning the procedure with $\varphi_y = g$ or $\varphi_z = h$ maybe be easier in some cases.

Fundamental Theorem for Line Integrals

Let $\mathbf{F}$ be a continuous vector field on an open connected region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$). There exists a potential function φ with $\mathbf{F} = \nabla\varphi$ (which means that $\mathbf{F}$ is conservative) if and only if

$$\int\limits_C \mathbf{F} \cdot \mathbf{T}\, ds = \int\limits_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A) \tag{133}$$

for all points A and B in R and all smooth oriented curves C from A to B.

Line Integrals on Closed Curves

Let R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) be an open region. Then $\mathbf{F}$ is a conservative vector field on R if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed smooth oriented curves C in R.

15.4 Green's Theorem

Let C be a simple closed smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA. \tag{134}$$

Two-Dimensional Curl

The **two-dimensional curl** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$. If the curl is zero throughout a region, the vector field is said to be **irrotational** on that region.

Area of a Plane Region by Line Integrals

Under the conditions of Green's Theorem, the area of a region R enclosed by a curve C is

$$\oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2} \oint_C (x\,dy - y\,dx) \tag{135}$$

Green's Thoerem, Flux Form

Let C be a simple closed smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\oint_C \mathbf{F} \cdot \mathbf{n}\,ds = \oint_C f\,dy - g\,dx = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) \tag{136}$$

where $\mathbf{n}$ is the outward unit normal vector on the curve.

Two-Dimensional Divergence

The **two-dimensional divergence** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$. If the divergence is zero thoughout a region, the vector field is said to be **source free** on that region.

15.5 Divergence and Curl

Divergence of a Vector Field

The **divergence** of a vector field $\mathbf{F} = \langle f,\, g,\, h\rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \tag{137}$$

If $\nabla \cdot \mathbf{F} = 0$, the vector field is **source free.**

Divergence of Radial Vector Fields

For a real number p, the divergence of the radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|r|^p} = \frac{\langle x,\, y,\, z\rangle}{(x^2+y^2+z^2)^{\frac{p}{2}}} \text{ is } \nabla \cdot \mathbf{F} = \frac{3-p}{|\mathbf{r}|^p} \tag{138}$$

Curl of a Vector Field

The **curl** of a vector field $\mathbf{F} = \langle f,\, g,\, h\rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\nabla \times \mathbf{F} = \operatorname{curl}\mathbf{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{\mathbf{k}} \tag{139}$$

If $\nabla \times \mathbf{F} = \mathbf{0}$, the vector field is **irrotational**.

Curl of a Conservative Vector Field

The **general rotation vector field** is $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where the nonzero constant vector $\mathbf{a} = \langle a_1,\, a_2,\, a_3\rangle$ is the axis of ration and $\mathbf{r} = \langle x,\, y,\, z\rangle$. For all nonzero choices of $\mathbf{a}$, $|\nabla \times \mathbf{F}| = 2|\mathbf{a}|$ and $\nabla \cdot \mathbf{F} = 0$. The constant angular speed of the vector field is

$$\omega = |\mathbf{a}| = \frac{1}{2}|\nabla \times \mathbf{F}| \tag{140}$$

Curl of a Conservative Vector Field

Suppose that $\mathbf{F}$ is a conservative vector field on an open region D of $\mathbb{R}^3$. Let $\mathbf{F} = \nabla\varphi$, where φ is a potential function with continuous second partial derivatives on D. Then $\nabla \times \mathbf{F} = \nabla \times \nabla\varphi = \mathbf{0}$; that is, the curl of the gradient is the zero vector and $\mathbf{F}$ is irrotational.

Divergence of the Curl

Suppose that $\mathbf{F} = \langle f,\, g,\, h\rangle$, where $f,\ g,$ and h have continuous second partial derivatives. Then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$: The divergence of the curl is zero.

Product Rule for the Divergence

Let u be a scalar-valued functino that is differentiable on a region D and let $\mathbf{F}$ be a vector field that is differentiable on D. Then

$$\nabla \cdot (u\mathbf{F}) = \nabla u \cdot \mathbf{F} + u(\nabla \cdot \mathbf{F}) \tag{141}$$

Properties of a Conservative Vector Field

Let $\mathbf{F}$ be a conservative vector field whose components have continuous second partial derivatives on an open connected region D in $\mathbb{R}^3$. Then $\mathbf{F}$ has the following equivalent properties.

1. There exists a potential function φ such that $\mathbf{F} = \nabla\varphi$
2. $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points A and B in D and all smooth oriented curves C from A and B.
3. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple smooth closed oriented curves C in D.
4. $\nabla \times \mathbf{F} = \mathbf{0}$ at all points of D.

15.6 Surface Integrals

Surface Integrals of Scalar-Valued Functions on Parameterized Surface

Let f be a continuous function on a smooth surface S given parametrically by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, where $R = \{(u, v) : a \leq u \leq b, c \leq v \leq d\}$. Assume also that the tangent vectors $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\rangle$, and $\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\rangle$ are continuous on R and the normal vectors $\mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R. Then the **surface integral** of the scalar-valued function f over S is

$$\iint_S f(x, y, z)\, dS = \iint_R f(x(u, v), y(u, v), z(u, v))|\mathbf{t}_u \times \mathbf{t}_v|\, dA \tag{142}$$

If $f(x, y, z) = 1$, the integral equals the surface area of S.

Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces

Let f be a continuous function on a smooth surfaces S given by $z = g(x, y)$, for (x, y) in a region R. The surface integral of f over S is

$$\iint_S f(x, y, z)\, dS = \iint_R f(x, y, g(x, y))\sqrt{z_x^2 + z_y^2 + 1}\, dA \tag{143}$$

If $f(x, y, z) = 1$, the surface integral equals the area of the surface.

Surface Integral of a Vector Field

Suppose $\mathbf{F} = \langle f, g, h\rangle$ is a continuous vector field on a region of $\mathbb{R}^3$ containing a smooth oriented surface S. If S is defined parametrically as $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, for (u, v) is a region R, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v)\, dA \tag{144}$$

where $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\rangle$ and $\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\rangle$ are continuous on R, the normal vector $\mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R, and the direction of $\mathbf{n}$ is consistent with the orientation of S. If S is defined in the form $z = g(z,\, y)$, for $(x,\, y)$ in a region R, then

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_R (-f z_x - g z_y + h)\, dA \tag{145}$$

15.7 Stokes' Theorem

Let S be a smooth oriented surface in $\mathbb{R}^2$ with a smooth closed boundary C whose orientation is consistent with that of S. Assume that $\mathbf{F} = \langle f, g, h \rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS \tag{146}$$

where $\mathbf{n}$ is the unit vector normal to S determined by the orientation of S.

Curl $\mathbf{F} = \mathbf{0}$ Implies $\mathbf{F}$ is Conservative

Suppose that $\nabla \times \mathbf{F} = \mathbf{0}$ throughout an open simply connected region D of $\mathbb{R}^3$. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed simple smooth curves C in D and $\mathbf{F}$ is a conservative vector field on D.

15.8 Divergence Theorem

Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D in $\mathbb{R}^3$ enclosed by a smooth oriented surface S. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV \tag{147}$$

where $\mathbf{n}$ is the unit outward normal vector on S.

Divergence Theorem for Hollow Regions

Suppose the vector field $\mathbf{F}$ satisfies the conditions of the Divergence Theorem on a region D bounded by two smooth oriented surfaces S_1 and S_2, where S_1 lies within S_2. Let S be the entire boundary of $D(S = S_1 \cup S_2)$ and let $\mathbf{n_1}$ and $\mathbf{n}_2$ be the outward unit normal vectors for S_1 and S_2, respectively. Then

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iint \mathbf{F} \cdot \mathbf{n} dS = \iint_{S_2} \mathbf{F} \cdot \mathbf{n_2}\, dS = \iint_{S_1} \mathbf{F} \cdot \mathbf{n_1}\, dS \tag{148}$$

MATH3304

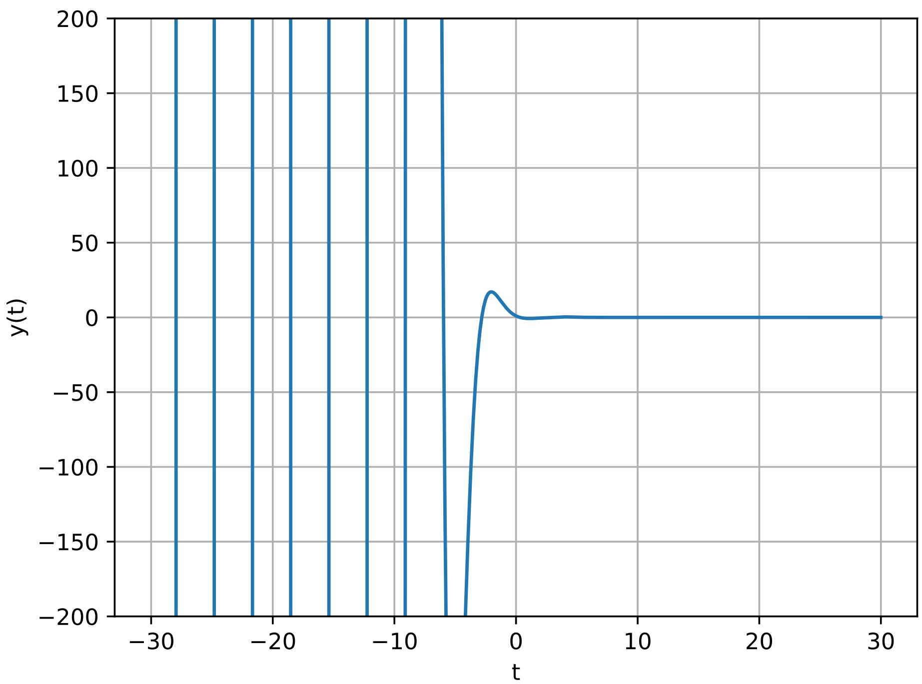
Differential Equations

Homework Graphs

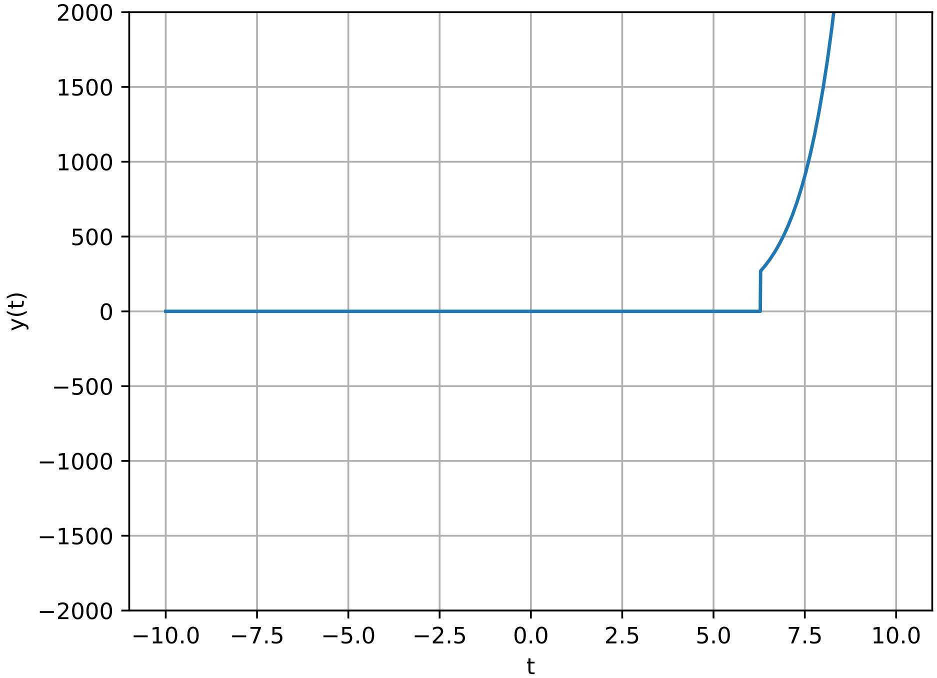
Illya Starikov

April 11, 2017

6.6 Problem #1



6.6 Problem #7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PHIL3225

Business Ethics

Business Portfolio

Illya Starikov

Due Date: May 10th, 2017

Word count: **1,339 words**.

1 Nature of Your Company

Currently, in the United States, one of the most controversial topics is self-driving cars. Google has notoriously been famous in the field for producing one of the earliest — and most famous — self-driving cars. This paved the way for Tesla to take said cars into the mainstream, with their Autopilot functionality. This made other companies take notice.

Since Tesla has shipped Autopilot in October of 2015, BMW, Mercedes Benz, Ford, and others have announced plans to create a fully-autonomous self-driving car. And why would a company not join the initiative? At the time of this writing, there are there are approximately 30,000 deaths in the United States *every year* caused by drivers. And these are not accidents where neither driver is to blame — most of these accidents **are human error**. By joining the fight to create a self-driving car, I would actually be fighting to save human lives — not just a few lives, but *saving millions of lives.*

Sidestepping the fight for humanity, my self-driving car business *will* become indispensable to humanity — as someone who has rode in a self-driving car, I can vouch for the utility. Getting over the initial hump that one is no longer in control (usually taking about 5 minutes), sitting back and *actually enjoying* driving is incomparable. Aside from the relaxation, my self-driving car company would provide much more utility, such as:

- Reduce traffic by preventing accidents and being able to know when to accelerate and decelerate in congested traffic

- Do mundane errands, like pick up groceries, people, etc.
- Virtually have no need for car insurance, saving the consumer hundreds-thousands of dollars a year

For reasons listed above, marketing the product is incredibly easily – so much so, advertising will not be necessary. Taking the Tesla approach (not spending at all on advertising) would actually be a better strategy for my company. At this point in the self-driving car life cycle, advertising would only cause a loss in focus and add additional expense to software. The same philosophy would initially apply to stakeholders. The self-driving car business would have too much politics for outside influence, so it would be ideal to be a private company with just initial funding from angel investors.

However, in a juxtaposition with that, everyone would *technically* have a stake in the company. People will come to rely on this technology; however, it *is dangerous*. A 2-ton machine made of aluminum going over 80 kph is nothing to scoff at, **it will kill you**. Because so many lives are at stake, it's imperative to make sure the algorithm works perfectly.

When considering my personal business, I would solely work on the software of the self driving car. Disconnecting from the hardware (except for the camera sensors) allows for actual focus on the brain (i.e., the algorithm). By abstracting the actual mechanics of the car, it would allow the company to focus on the major factors that matter. This would include:

- The specialized artificial intelligence. This would actually determine when to break, where to turn, how far ahead to stop, what to actually do.
- The visualization and camera sensors. This would mean taking in live video and able to analyze what is happening.
- The quality assurance. Again, there will be people at risk. It *has to be perfect*, so a dedicated team to make sure the code is flawless would be necessary.

Taking these into consideration, the team size would have to be *large* – a guess would be at least 250-500. It's better to think the size of the teams as percentages. Roughly 70% of the team would be solely dedicated to engineering the algorithm, 10% for the cameras and sensors, and the 20%

leftover would work on quality assurance. The majority 80% would simply be programmers, each with dedicated skills in

- Artificial Intelligence and Machine Learning
- Camera Input/Outputs
- Code Architecture

2 Moral Dilemmas

When considering the company, three major moral dilemmas arise:

1. The Trolly-esque problem
2. The "I am taking millions of jobs" problem
3. The luddite problem

2.1 The Trolly-esque

Consider the trolly problem

> Suppose you are standing on train tracks. You notice on one side of the tracks, there is an oncoming trolley. On the other side, there is a fork in the tracks. On one side, there is a person tied to the tracks; on the other, five people are tied. The tracks are oriented towards the five people; however, there is a lever to divert the tracks. Do you save five at the expense of one?

This exactly applies to self driving cars. If one is driving, there might come a situation where there is a brake failure, and the artificial intelligence in the car might have to either kill the one passenger or the five pedestrians. This will be at the hands of the programmer to decide.

Is it the responsibility of the car to protect the driver, or take a utilitarian point and kill the driver for the five pedestrians? These are decisions I, the CEO, will have to make on behalf of the self driving cars and my company.

2.2 The "I Am Taking Millions of Jobs" Problem

It is no secret that self driving cars will take over *millions* of jobs; this involves many chauffeurs, taxi drivers, truckers, and more.

At the hand of my company, people who have to support their families will be out of a job because of my company.

2.3 The Luddite Problem

As they did with the cotton mill, luddites will definitely try to prevent self-driving car wave. Unfortunately, unlike the era of the cotton mill, many luddites will also take the form of congressmen; furthermore, not only will there be a lot of opposition from the house and senate, but other countries as well. Politically, there will be a huge barrier for me to overcome.

3 Moral Solutions

Below I will list the solutions to the above problem

3.1 The Trolly-esque Solution

I always hold the philosophy that the simplest solution is *generally* the best one. And that is a philosophy I will apply here. In reality, this situation will arise so little of the time (in a electric car, with about 10 moving parts, break failure is very uncommon), that I would not even program the car to do anything explicitly. Programming the car to protect the driver adds unnecessary code. The car will do a calculation of all methods of course correction, and whatever is the most optimal, I will force it to chose.

However, **I would never make this known**. When asked how my car handles this problem, I will never comment. If I say the car chooses the driver, then the argument could be made that I choose one life over five. If I say it chooses the five people, people would be scared to use the car. It is a lose-lose situation.

3.2 The "I Am Taking Millions of Jobs" Solution

There is no *real* good solution to this. This is an intrinsic problem of the self driving car business. Although I would creating a couple hundred jobs, it will take millions from hard working people. I would have to hope the government could create a transition system, and possibly lobby for it. But as just what my company could do, there is next to nothing.

3.3 The Luddite Problem

This is also an intrinsic problem of the company, however more easily fixed. The choice is obvious: jobs are replaceable, human life is not. So self driving cars *should* be accepted by the government.

The best solution would be to create a coalition with other companies to fight the government's stance on self driving cars. I feel it is a moral duty, not just in a Kantian and a Utilitarian context, but in a *these save lives* context. As someone who has almost lost his life and his mother's life in a car wreck, I feel it is my duty to create market adoption.

PHYS1135

Physics I

0.1 State Variable

$C_v is specific heat for constant volume.$

$p,\ V,\ T,\ n,\ U$. Cannot write Q, because you can pull heat in and out. The relation ships of these:

$$\begin{aligned} pV &= nRT \\ U &= \frac{3}{2}nRT \\ &= N_{molecules}E_{kinetic} \end{aligned}$$

We can describe this in a pV diagram. When you see a pV diagram that has an area under the curve, you can can calculate it with:

$$W = \int pdv$$

How this is derived:

$$\begin{aligned} W &= \int Fdv \\ &= Ap\frac{dv}{A} \\ &= \int pdv \end{aligned}$$

Now talking about non state variable:

$$U = Q - W \qquad (1)$$

0.2 Processes

isobaric $p =$ constant

isochoric $V =$ constant

isothermic $T =$ constant

adiabatic $Q =$ constant

To maintain the same pressure (isobaric), apply heat and increase temperature. To maintain the same volume (isochoric), cool it down. To keep the temperature the same (isothermal), keep it in thermodynamic equilibrium while changing the pressure / volume via a heat bath. To keep the heat the same, insulate the system.

1. Isobaric.
   - $W = \int pdV = p\Delta V = nR\Delta T.$
   - $Q = nC_p\Delta T$
   - $\Delta U = Q - W = n(c_p - R)\Delta T = nc_v\Delta T$
2. Isochoric.
   - $W = \int pdV = 0$
   - $Q = nc_v\Delta T$
   - $\Delta U = Q = nc_v\Delta T = \frac{3}{2}nRT \rightarrow c_v = \frac{3}{2}R$
3. Iosthermal.
   - $W = \int pdV = \int_{vi}^{vf}(nRT)dV = nRTln(\frac{V_f}{V_i}$
   - $\Delta U = 0 \rightarrow Q - W = 0 \rightarrow Q = W$
4. Adiabatic
   - $\Delta Q = 0 \rightarrow \Delta U = -\Delta W \rightarrow du = -dW$
   - Just look in lecture notes for derivation.
   - $pV^{\gamma} = pV^{\frac{c_p}{c_v}}$ constant

PHYS2135

Physics II

1 Electric Charge, Coulomb's Law, Electric Field, Motion of a Charge in Electric Field

1.1 Book Notes

- When charges are at rest in our frame of reference, they exert electrostatic forces on each other.
- Electrostatic forces are governed by a simple relationship known as Coulomb's law and are most conveniently described by using the concept of electric field.
- Two positive charges or two negative charges repel each other. A positive charge and a negative charge attract each other.
- The algebraic sum of all the electric charges in any closed system is constant.
- The magnitude of charge of the electron or proton is a natural unit of charge.
- The electric force on a charged body is exerted by the electric field created by other charged bodies.

1.2 Lecture Notes

- Law of conservation of charge: the net amount of electric charge produced in any process is zero.

1.3 Recitation

- Office is 115
  - Tue/Thu: 10-Noon, 2-4:30
  - Wed: 8-2, 2-4:30
  - Fri: 8-Noon

2 Electric Field of a Charge Distribution

2.1 Lecture Notes

- $\lambda \neq$ to wavelength, but linear charge density (charge per length).
- Arc length is angle (radians) times radius ($S = r\theta$).
- No two field lines can cross.

2.2 Recitation

- Because in nature, point charges are not common, we do a distributed charges.
  - We do this via superposition (i.e. add up vectorily to get the total charge).
- Suppose you have a rod, and an origin at the center, and a point h distance at the origin.
  - $dE_x = dE \sin\theta$
  - Likewise, $dE_y = dE \cos\theta$

$$dE_y = dE \cos\theta \quad (1)$$
$$= \frac{k\lambda dx}{r^2}\frac{h}{r} \quad (2)$$

- This is one of two charge distribution we often encounter.
  - Still the same process.

3 Electric Field Lines, Electric Dipoles, Electric Flux, Gauss' Law

3.1 Book Notes

- The direction of $\tilde{\mathbf{p}}$ is from the negative to the positive charge.

3.2 Lecture Notes

- The electric field always depends on qd.
  - dipole moment vector $\vec{p} = q\vec{d}$
  - Torque on the dipole is exactly the same as classical mechanics.
- Remember, zero potential energy does not mean minimum potential energy!
- The **electric flux** passing through a surface is the number of electric field lines that pass through it.
- For a closed surface, dA is normal to the surface and always points away from the inside.
- The electric field is a vector field, so a constant electric field is one that does not change with position or time.
- If a conductor is in electrostatic equilibrium, any excess charge must lie on its surface, so for the charge to be uniformly distributed throughout the volume, the object must be an insulator.

3.3 Recitation

- Electric field lines just give an easy way to imagine what the force would be.
- The distance $\vec{\mathbf{d}}$ points from the negative to the positive.
  - That where the dipole moment comes from.
- $U = -\vec{p}\vec{E}$
- $\phi_E = \oint \vec{E} \cdot d\vec{A}$
  - Area vector points outward.

$$
\begin{aligned}
\phi_E &= \oint \vec{E} \cdot d\vec{A} && && (1) \\
&= \oint E \; da \cos\theta && \text{Cross product expansions} && (2) \\
&= E \oint dA && \text{If E is constant, pull it out} && (3) \\
&= E \times \text{Surface Area} && && (4) \\
& && && (5)
\end{aligned}
$$

4 Gauss' Law Calculations, Conductors and Electric Fields

4.1 Book Notes

- Whether there is a net outward or inward electric flux through a closed surface depends on the sign of the enclosed charge.
- Charges outside the surface do not give a net electric flux through the surface.
- The net electric flux is directly proportional to the net amount of charge enclosed within the surface but is otherwise independent of the size of the closed surface.
- Gauss's law states that the total electric flux through any closed surface (a surface enclosing a definite volume) is proportional to the total (net) electric charge inside the surface.
- When excess charge is placed on a solid conductor and is at rest, it resides entirely on the surface, not in the interior of the material.
- Electrostatic equilibrium means there is no net motion of tne charges inside the conductor.
  - The electric field inside the conductor must be zero.
  - If this were not the case, charges would accelerate.
- Any excess charge must reside on the outside surface of the conductor.
- The electric field just outside a charged conductor must be perpendicular to the conductor's surface.
- The magnitude of the electric field just outside a charged conductor is equal to $\frac{|\sigma|}{\epsilon_0}$, where $|\sigma|$ is the magnitude of the local surface charge density.

4.2 Lecture Notes

- The electric field inside the conductor must be zero.
  - If not, the system would accelerate.

4.3 Recitation

- We're doing some vector review.
  - $\vec{A} \cdot \vec{B} = AB \cos\theta$
    * max at $\theta = 0$
    * min at $\theta = \pi$

- $|\vec{A} \times \vec{B} = |AB \sin\theta|$
  - max at $\theta = \frac{\pi}{2}$
  - min at $\theta = 0$
- Back to physics. $\vec{F} = q\vec{E}$
- Inside conductor, $E = 0$. Otherwise electrons would still be moving.
  - This brings us to Gauss's law.
- A good way to calculate charge would be to start at the inner most surface and work on the way out.

5 Electric Potential, Electric Potential Energy

5.1 Book Notes

- The potential-energy difference $U_a - U_b$ equals the work that is done by the electric force when the particle moves from a to b. When U_a is greater than U_b, the field does positive work on the particle as it "falls" from a point of higher potential energy (a) to a point of lower potential energy (b).
- The potential-energy difference $U_a - U_b$ is then defined as the work that must be done by an external force to move the particle slowly from b to a against the electric force.
- **Potential** is potential energy per unit charge.
- SI unit of potential is called one volt (1**V**).
- The potential difference between two points is often called **voltage**.
- V_{ab}, the potential of a with respect to b, equals the work that must be done to move a UNIT charge slowly from b to a against the electric force.
- The electric potential at a certain point is the potential energy that would be associated with a unit charge placed at that point. That's why potential is measured in joules per coulomb, or volts. Keep in mind, too, that there doesn't have to be a charge at a given point for a potential V to exist at that point. (In the same way, an electric field can exist at a given point even if there's no charge there to respond to it.)
- Moving with the direction of $\tilde{\mathbf{E}}$ means moving in the direction of *decreasing* V, and moving against the direction of $\tilde{\mathbf{E}}$ means moving in the direction of *increasing* V.

5.2 Lecture Notes

- $\Delta U = -[W_{\text{conservative}}]_{i \to f}$
  - Always ask yourself which work you are calculating.
- Don't fall into the trap of making r_{12} a square.
- Potential energies are defined relative to some configuration of objects that you are free to choose.
  - For example, it often makes sense to define the gravitational potential energy of a ball to be zero when it is resting on the surface of the earth, but you don't have to make that choice.

- Our equation for the electric potential energy of two charged particles uses the convention that the potential energy is zero when the particles are infinitely far apart.
- Protons fall down, electrons fall up.
- An electron volt (eV) is the energy acquired by a particle of charge e when it moves through a potential difference of 1 volt.

6 Electric Potentials of Charge Distributions, Equipotentials, Potential Gradient

6.1 Book Notes

- An equipotential surface is a three-dimensional surface on which the electric potential V is the same at every point.
- On a given equipotential surface, the potential V has the same value at every point. In general, however, the electric-field magnitude E is not the same at all points on an equipotential surface.
- When all charges are at rest, the surface of a conductor is always an equipotential surface
- In an electrostatic situation, if a conductor contains a cavity and if no charge is present inside the cavity, then there can be no net charge anywhere on the surface of the cavity.
- surface charge density on the wall of the cavity is zero at every point.
- Don't confuse equipotential surfaces with the Gaussian surfaces we encountered in Chapter 22. Gaussian surfaces have relevance only when we are using Gauss's law, and we can choose any Gaussian surface that's convenient. We cannot choose equipotential surfaces; the shape is determined by the charge distribution.

6.2 Lecture Notes

- The disk of charge will be on exam.
- The "Ed" equation does not require rectangular plates, or any plates at all. It works as long as E is uniform and parallel or antiparallel to d.

6.3 Recitation

- Potential is a scalar, which awesommmmee.
  - But we can't exploit symmetry because of no directions.

7 Capacitance, Capacitors in Series and Parallel

7.1 Book Notes

- Any two conductors seperated by an insulator (or a vacuum) form a capacitor.
- Don't confuse the symbol C for capacitance (which is always in italics) with the abbreviation C for coulombs (which is never italicized).
- Thus capacitance is a measure of the ability of a capacitor to store energy.
- The reciprocal of the equivalent capacitance of a series combination equals the sum of the reciprocals of the individual capacitances.
- The magnitude of charge is the same on all plates of all the capacitors in a series combination; however, the potential differences of the individual capacitors are not the same unless their individual capacitances are the same. The potential differences of the individual capacitors add to give the total potential difference across the series combination: $V_{\text{total}} = V_1 + V_2 + V_3 + \cdots$
- The equivalent capacitance of a parallel combination equals the sum of the individual capacitances.
- The potential differences are the same for all capaci- tors in a parallel combination; however, the charges on individual capacitors are not the same unless their individual capacitances are the same. The charges on the individ- ual capacitors add to give the total charge on the parallel combination: $Qtotal = Q1 + Q2 + Q3 + \cdot$

7.2 Recitation

- The simplest capacitor is just two plates, both with a charge $\pm Q$ (they different).
  - Capacitance, by definition, has to be positive.
  - $C = \frac{\sigma A \epsilon_0}{\sigma d} = \frac{\kappa \epsilon_0}{d}$
    - Only depends on the geometry.
- *Does example from lecture, except with sphere..*
  - Except specifies direction with $\hat{r}$
- With series, $Q_1 = Q_2$.
  - $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}$

- Opposite is true
  - $V_1 = V_2$
  - $C_{eq} = C_1 + C_2 + \cdot + C_n$

9 Energy stored in Capacitors and Electric Fields; Dielectrics

9.1 Lecture Notes

- Charge goes up (conceptual example).
- Lets you apply higher voltages (so more charge).
- Lets you place the plates closer together (make d smaller).
- Increases the value of C because $\kappa > 1$.

9.2 Recitation

- Before the exam, we had a capacitance equation with a κ.
  - Suppose we disconnect from battery, so Q is constant¿
  - Now put an insulator inside capacitor.
    - Tada, there's you κ
    - Insulator means electrons are stuck where they're at.
    - $\kappa > 1$, usually.
  - This causes an electric field inside, causes a polarization.
- We can express the energy in three different forms, depending on what we know about the system.
- Energy conservation is inapplicable to these problems.

10 Electric Current; Current Density; Resistance

10.1 Book Notes

- An electric current consists of charges in motion from one region to another. If the charges follow a conducting path that forms a closed loop, the path is called an electric circuit.
- In electrostatic situations (discussed in Chapters 21 through 24) the electric field is zero everywhere within the conductor, and there is no current. However, this does not mean that all charges within the conductor are at rest.
- Although we refer to the direction of a current, current as defined by Eq. (25.1) is not a vector quantity. In a current-carrying wire, the current is always along the length of the wire, regardless of whether the wire is straight or curved. No single vector could describe motion along a curved path. We'll usually describe the direction of current either in words (as in "the current flows clockwise around the circuit") or by choosing a current to be positive if it flows in one direction along a conductor and negative if it flows in the other direction.
- Current density $\vec{J}$ is a vector, but current I i snot — the difference is that the current density $\vec{J}$ describes how charges flow at a certain point, and the vector's direction tells you about the direction of the flow at that point.
- The greater the resistivity (ρ), the greater the field needed to cause a given current density, or the smaller the current density caused by a given field.

10.2 Lecture Notes

- Current goes down to milli-Amps — **Remember for test**.
- Current is a scalar — but can be negative.

10.3 Recitation

- Ohm's Law, in general form, $\vec{J} + \sigma\vec{E}$
- In this class, Ohm's law will just be current though a wire.
- We can assume the current density is proportional $\frac{I}{A}$
- The electric field should be uniform!
- $V = IR$ servers as our definition of resistance
- Some materials actually have a $-\alpha$

11 EMF; Electric Power

11.1 Book Notes

- For a conductor to have a steady current, it must be part of a path that forms a closed loop (or a complete circuit).

- A battery is not a current source — You might have thought that a battery or other source of emf always produces the same current, no what circuit it's used in. $\mathcal{E} - Ir = IR \implies I = \frac{\mathcal{E}}{R+r}$ says it isn't so. The greater the resistance R of the external circuit the less the current the source will produce.

11.2 Recitation

- Rate of energy conversion: $P = I\mathcal{E}$

- Energy lost $P = I^2 r_1$

- Total power supplied $P = I\mathcal{E}_1 - I^2 r_1$

12 Resistors in Series and Parallel, Kirchoff's Rules

12.1 Book Notes

For capacitors in series,

$$V_{ax} = IR_1 \qquad V_{xy} = IR_2 \qquad V_{yb} = IR_3 \tag{1}$$

$$V_{ab} = V_{ax} + V_{xy} + V_{yb} = I(R_1 + R_2 + R_3) \tag{2}$$

$$\frac{V_{ab}}{I} = R_1 + R_2 + R_3 \tag{3}$$

$$R_{eq} = R_1 + R_2 + R_3 \tag{4}$$

A similar argument can be made for resistors in parallel, except I is unknown, which add.

- A junction in a circuit is a point where three or more conductors meet. A loop is any closed conducting path.

13 l

13.1 Book Notes

13.2 Lecture Notes

13.3 Recitation

- The magnetic field is directly analogous to the electric field
  - However, the fields lines are circular (because no monopoles)
  - The force is cross product

Exam _num$_1$*QuickReference*

Illya Starikov

August 7, 2025

1 Prefix

Factor	**Name**	**Symbol**
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f

Test II Study Guide

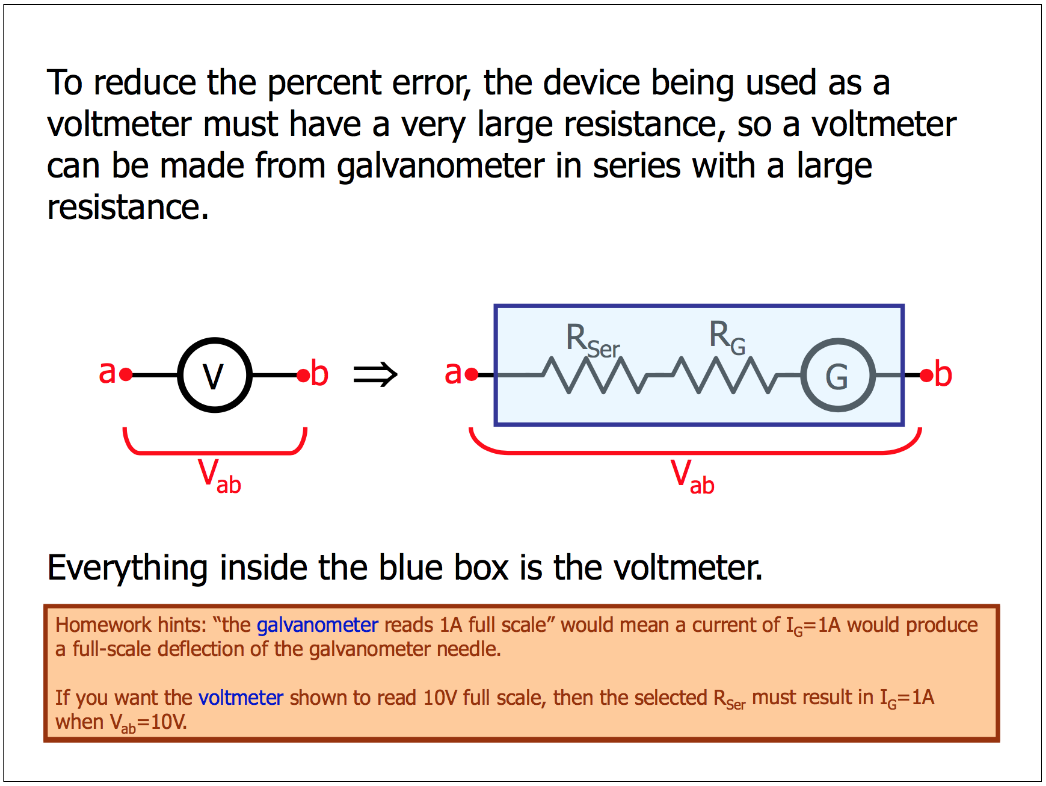
Illya Starikov

August 7, 2025

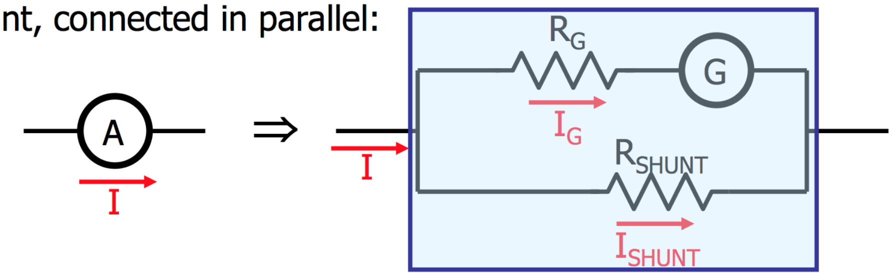
1 Notes

- If a dielectric fills one half a the space between plates, it's the same as two capacitors in parallel — one with a dielectric, one not.
- Current is in the direction of flow of **positive charge**.
- Materials that are ohmic have a linear I vs. V graph.
  - Anything else (like quadratic) are nonohmic.
- For resistors **in series**,
  - $R_{eq} = \sum_i R_i$ (opposite of capacitors)
  - Currents I is the same (same as capacitors)
  - V's add (same as capacitors)
- For resistors **in parallel**,
  - $\frac{1}{R_{eq}} = \sum_i \frac{1}{R_i}$ (opposite of capacitors)
  - Currents I is the same (same as capacitors)
  - V's add (same as capacitors)
- Power $= \frac{\text{Energy Transformed}}{\text{Time}}$
- Kirchhoff's Junction Rule: at any junction point, the sum of all currents entering the junction must equal the sum of all currents leaving the junction.

- Kirchhoff's Loop Rule: the sum of the changes of potential around any closed path of a circuit must be zero.



A galvanometer-based ammeter uses a galvanometer and a shunt, connected in parallel:



Everything inside the blue box is the ammeter.

The resistance of the ammeter is

$$\frac{1}{R_A} = \frac{1}{R_G} + \frac{1}{R_{SHUNT}}$$

$$R_A = \frac{R_G R_{SHUNT}}{R_G + R_{SHUNT}}$$

- For charging a circuit, $I(t) = \frac{\varepsilon}{R} e^{-\frac{t}{RC}}$
- For discharging a circuit, $I(t) = I_0 e^{-\frac{t}{RC}}$
  - I_0 for charging is equal to I_0 for discharging only if the discharging capacitor was fully charged.
- In a series RC circuit, the same current I flows through both the capacitor and the resistor. Sometimes this fact comes in handy.
- Ohm's law **only** applies to resistors, not capacitors.
- Magnetic field lines point away from north, towards south.
  - Same notation as electric fields!
- In a uniform magnetic field, force is always radially outward. Therefore, $a = {}^{v^2}/_{r}$.

- Period $T = \frac{2\pi r}{v}$. (However, easier to remember distance = velocity$\cdot$ time $\implies T = \frac{\text{distance}}{\text{velocity}}$)
- frequency $= \frac{1}{T}$

2 Units

- Resistance: $1\Omega = \frac{1V}{1A}$
- Resistivity: $\rho = \Omega \cdot m$
- Magnetic field: $1T = \frac{1\,\text{kg}}{\text{C}\cdot\text{s}}$
- Older unit: $1\,\text{G} = 10^{-4}\text{T}$

3 Unit Prefixes

10^{-15}	femto (f)
10^{-12}	pico (p)
10^{-9}	nano (n)
10^{-6}	micro (μ)
10^{-3}	milli (m)
10^{-2}	centi (c)
10^{-1}	deci (d)

PHYS2305

Modern Physics

Modern Physics Review

Illya Starikov

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1 Special Relativity

From relativity, we know

1. All inertial reference frames are equivalent.
2. The speed of light is the same in all inertial reference frames.

From this, we notice that

1. Time and space depend on velocity ($\vec{v}$).
2. Moving clocks appear to run slow.
3. The length of a moving object, in the direction of motion, will appear shorter.

Eloquently, this can be described as

$$t = \gamma t_0 \qquad L = \frac{L_0}{\gamma}$$

Where $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$, t is the time according to the outside observer, t_0 to be proper time (i.e. the time measured by the moving object), L is length to outside observer, and L_0 is proper length.

2 Energy and Momentum

Relativistic energy and momentum can be defined as

$$E = \gamma m_0 c^2 \qquad \vec{p} = \gamma m_0 \vec{v}$$

From the this, we can derive the more fundamental equation:

$$E^2 = m_0^2 c^4 + p^2 c^2 \tag{1}$$

For kinetic energy, we can eloquently describe it as $(\gamma - 1)m_0 c^2$, this is simply the rest mass energy (mc^2) subtracted from the total energy (γmc^2). At low speeds (i.e. $v \ll c$), we can simply use a Taylor Series expansion to get $E \approx {}^1\!/_2 m_0 v^2 + \frac{3m_0v^4}{8c^2} + \cdots$. Ignoring the other terms, we get $E \approx {}^1\!/_2 mv^2$, classical energy!

If we take m_0 to be 0, this implies $E = pc$ (From Equation 1). Using de Broglie wavelength ($\lambda = {}^h\!/_p$), this is implies $E = h\nu$. This can be combined with the fact that $\lambda\nu = c$, which allows many permutations of the equations. We can also show that the kinetic energy $KE = \frac{p^2}{2m}$ To summarize, for $m_0 = 0$,

$$E = h\nu = \frac{hc}{\lambda} = pc$$

3 The Three Experiments

There were three experiments done to prove the quantum nature of light and particles.

3.1 The Photoelectric Effect

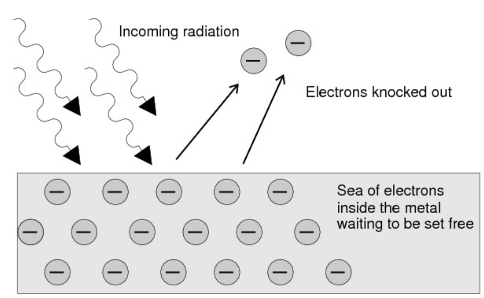

Figure 1 – The Photoelectric Effect.

Suppose we have a sea of electrons within a metal. It takes some work to escape the sea, described by the work function Φ. We can relate the energy of the photon, the work function, and

the resulting energy of the electrons
by $h\nu = \Phi + KE_{\text{max}}$.

3.2 Compton Scattering

We can scatter a photon off an electron, inelastically, and after some tedious math we can realize $\lambda_2 - \lambda_1 = \frac{h}{m_0 c}(1 - \cos\theta)$, where λ_2 is the wavelength after scattering, λ_1 is the wavelength before the scattering, and m_0 the electron rest mass. From this we realize that there is a decrease in the energy of the photon, resulting in an increase of wavelength, which we know as the Compton effect.

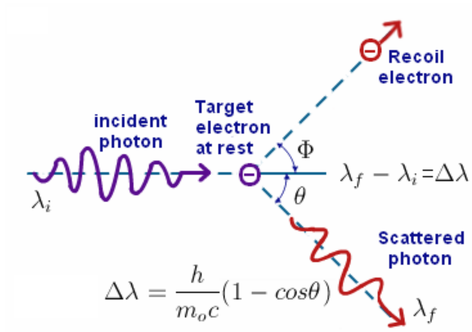

Figure 2 – Compton scattering.

3.3 Blackbody Radiation

Blackbody radiation refers to an object or system which absorbs all radiation incident upon it and re-radiates it. This effect can be characterized by the radiating system alone; it does not depend on the type of radiation incident upon it. The radiated energy can be considered to be produced by standing wave or resonant modes of the cavity which is radiating. The major effect of this is that the modes must be quantized. We can define the energy per unit volume per unit frequency

$$U(\nu) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{\frac{h\nu}{kT}} - 1}$$

where k is the Boltzmann constant and T is the absolute temperature of the body. The entity $\frac{h\nu}{e^{h\nu/kT}-1}$ is the average energy per mode, and $\frac{8\pi\nu^2}{c^3}$ counts the number of modes available.

4 Wave Nature of Massive Particles

$$|\psi(x)|^2 dx = \text{the probability of finding the particle in the range of } x \text{ to } x + dx$$

Because $|\psi(x)|^2$ is a continuous probability distribution, we must normalize the wavefunction so that the probability has to add up to 100%,

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1$$

4.1 Uncertainty Principle

It can be derived that for an particular measurement,

$$\Delta t \Delta E \geq \frac{\hbar}{2} \qquad \hbar = \frac{h}{2\pi}$$

Or, more famously,

$$\Delta x \Delta p \geq \frac{\hbar}{2} \qquad \hbar = \frac{h}{2\pi}$$

4.2 Infinite Potential Well

Imagine a potential wall where the wall go to infinity, with a distance L between the walls. Our wave function ψ must have the boundary conditions $\psi(x = 0) = 0$ and $\psi(x = L) = 0$. From this we realize we can only fit half-wavelengths of the wave into the box; that is, $n\lambda/2 = L$. To calculate the energy,

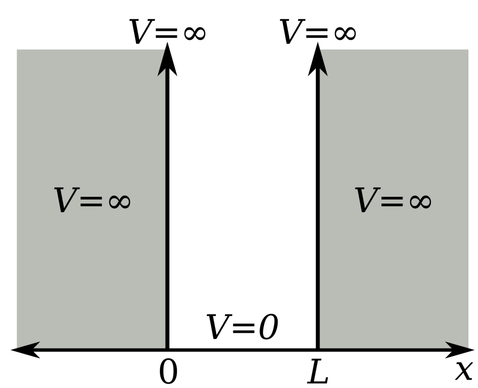

Figure 3 – Particle in a box (or infinite potential well).

$$E = \frac{p^2}{2m} = \frac{n^2h^2}{8mL^2}$$

because $\lambda = \frac{2L}{n}$ and $p = \frac{h}{\lambda}$.

4.3 Wave Motion

We know we can describe just about any wave by $\cos(kx - \omega t)$, where $k = \frac{2\pi}{\lambda}$ and $\omega = 2\pi\nu$. Consequently, $p = {}^{h}/_{\lambda} = \hbar k$ and $E = h\nu = \hbar\omega$.
Furthering our wave mathematics, we can consider the phase velocity as

$$v_p = \frac{\omega}{k} = \frac{E}{p} = \nu\lambda$$

When considering a group, we can calculate the velocity of the group (or the packet), v_g, as

$$v_g = \frac{\partial\omega}{\partial k} = \frac{\partial E}{\partial p}$$

4.4 Hydrogen Atom

As we have proven with our three experiments, energy states are quantized. This implies that the energy levels in an atom come in integer levels (i.e. energy level $n = 1, 2, 3, \ldots, \infty$, where ∞ is ionization).

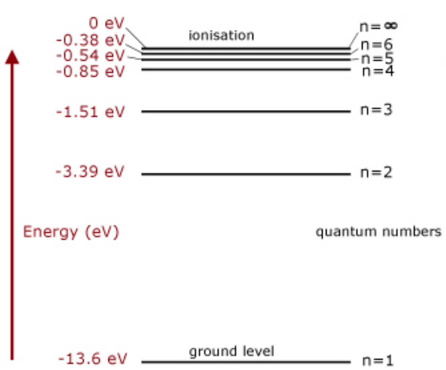

Figure 4 – Energy levels of hydrogen atom.

From this, we can determine that the energy at any level $E_n = \frac{E_1}{n^2}$, where $E_1 < 0$. For photo-absorption,

$$\begin{aligned} h\nu = E_f - E_i &= \frac{E_1}{n_f^2} - \frac{E_1}{n_i^2} \\ &= E\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \end{aligned}$$

Similarly, for emission,

$$\begin{aligned} h\nu = E_i - E_f &= \frac{E_1}{n_i^2} - \frac{E_1}{n_f^2} \\ &= E\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right) \end{aligned}$$

PSYC1101

Psychology 101

A Beautiful Mind

Illya Alexsandrovich Starikov

July 27, 2025

1 The Role of Psychology

The move is centered around John Nash, who, as find out later in the story, has schizophrenia. In the remainder of this movie we are taken through Nash's life as he copes with his hallucinations and illusions.

2 Main Character

John Nash experiences an aggressive case of schizophrenia. His extraordinary intelligence (aside from his contribution to economics, he also worked on complex manifolds and the Riemann hypothesis, two very difficult areas of mathematics) causes him to experience very vivid episodes of hallucinations. A possible reasoning for the different characters is a coping mechanism — the roommate for his loneliness, the child for his fear of being a new father, the cryptography for having to do something great.

3 Psychological Treatment

John Nash first received regular treatments of *Insulin shock therapy*, in which he was repeatedly injected with insulin to induce comas. Because this is not a sustainable form of treatment outside of medical facilities, he was then given a regular prescription of an anti-psychotic. However, the side effects were too detrimental to Nash's mathematical work and his love life, he decides to mentally confront it. Instead of blocking it out, he decides to accept his hallucinations.

Famous People Extra Credit

Illya Alexsandrovich Starikov

July 27, 2025

1. Charles Darwin
2. Dorothea Dix
3. Sigmund Freud
4. G. Stanley Hall
5. William James
6. Ivan Pavlov
7. Jean Piaget
8. Carl Rogers
9. B. F. Skinner
10. Sigmund Freud